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Structure theorems of H_4-Azumaya algebras

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Abstract

Let k be a field and H_4 be Sweedler's 4-dimensional algebra over k. It is well known that H_4 has a family of triangular structures R_t indexed by the ground field k and each triangular structure R_t makes the H_4-module category ${}_{H_4}\mathcal{M}$ a braided monoidal category, denoted ${}_{H_4}\mathcal{M}^{R_t}$. In this paper, we study the Azumaya algebras in the categories ${}_{H_4}\mathcal{M}^{R_t}$. We obtain the structure theorems for Azumaya algebras in each braided monoidal category ${}_{H_4}\mathcal{M}^{R_t}$, $t \in k$. Utilizing the structure theorems we obtain a scalar invariant for each Azumaya algebra in the aforementioned categories.
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0. Introduction

In [22], Wall introduced the Brauer–Wall group $\mathrm{BW}(k)$ of central simple graded algebras (CSGAs) over a field k generalizing the Brauer group of central simple algebras over k. The significance of the group $\mathrm{BW}(k)$ can be illustrated by its structure relations to quadratic forms, Clifford invariants, Hasse invariants and Witt invariants etc. The key to

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achieving these relations is a full understanding of the structure of a CSGA. Wall found that a CSGA is either of even type or of odd type according to the structure of its center. Thus he obtained a complete characterization of structures of both types (see [11,22]). Henceforth, the aforementioned invariants have been assigned to CSGAs according to their structure and their types. Wall's structure approach to the computation of the Brauer–Wall group has not yet been adopted to computing other generalized Brauer groups such as the Brauer–Long group $\mathrm{BD}(k, H)$ of a commutative and cocommutative Hopf algebra H (see [12]), or the equivariant Brauer group $\mathrm{BM}(k, H, R)$ of a quasitriangular Hopf algebra (H, R) (see [4]), mainly due to the difficulty of understanding the structure of an H-Azumaya algebra. In this paper, we adopt Wall's approach to investigating Azumaya algebras over Sweedler's 4-dimensional Hopf algebra H_4, which are also called differential super Azumaya algebras (simply DSA algebras), in this case the braided product is given by Wall's graded product (cf. [21]). A DSA algebra is nothing more than a CSGA with a graded differential. Thus Wall's structure theorems of CSGAs apply to our DSA algebras. Therefore we may restrict our attention to the actions of graded differentials of DSA algebras. We show that the differential of a DSA algebra A is uniquely determined by a homogeneous element of degree one if A does not represent an element in the subgroup $\mathrm{BW}(k)$ (in this case, the differential is said to be nontrivial). Based on this observation, we are able to obtain a full picture of the structure of a DSA algebra with a nontrivial differential (see Theorems 3.4 and 3.6). We obtain that a DSA algebra with a nontrivial differential is a product of a CSGA and a graded quadratic extension of k with a natural differential. In particular, we assign to each DSA algebra a scalar invariant. Two Brauer equivalent DSA algebras share the same scalar invariant. This explains why the Brauer group of DSA algebras is the direct product $\mathrm{BW}(k) \times k^+$, computed in [20], where k^+ is the additive group of k. It gives us a better understanding of the nontorsion subgroup k^+ generated by quaternion algebras. The group k^+ is in fact the group of scalar invariants of DSA algebras, and can be generated by graded quadratic extensions with a differential. In this sense, graded quadratic extensions play the role of "minimal generators" of the Brauer group of DSA algebras.

The paper is organized as follows. In Section 1, we recall the Yetter Drinfeld module algebras and Azumaya algebras over a Hopf algebra H. Cocycle twisting of a Hopf algebra will be discussed. Since the Hopf algebra we will deal with in this paper is Sweedler's 4-dimensional Hopf algebra H_4, we will present the cocycles and the triangular structures of H_4. A class of 2-dimensional nontrivial H_4-Azumaya algebras will be introduced. They are quadratic extensions of the ground field k with a natural H_4-action.

In Section 2, we prove the centralizer theorem in a general setting. Let H be a Hopf algebra with a bijective antipode, and A an H-Azumaya algebra. If B is an H-Azumaya subalgebra of A, then both the left centralizer $C^l_A(B)$ and the right centralizer $C^r_A(B)$ of B in A are H-Azumaya algebras. Moreover, $A \cong C^l_A(B) \# B \cong B \# C^r_A(B)$ as H-Azumaya algebras. The centralizer theorem will play a critical role in working out the structures of Azumaya algebras over H_4 in the next two sections.

In Section 3, we investigate the structures of H_4-Azumaya algebras in the category ${}_{H_4}\mathcal{M}^{R_0}$ of left H_4-modules endowed with the braided tensor product induced by the triangular structure R_0. Such an H_4-Azumaya algebra is either of even type or of odd type when considered as a graded Azumaya algebra. We show that in both cases, an H_4-Azumaya algebra is a graded product of a CSGA and a quadratic extension $k\langle\sqrt{\alpha};\beta\rangle$ of k with the H_4-action given in Section 1.5. In particular, we assign to each H_4-Azumaya algebra a scalar invariant. The set of scalar invariants forms the additive group k^+. Thus we know from the structure of any H_4-Azumaya algebra that the equivariant Brauer group $\mathrm{BM}(k, H_4, R_0)$ is generated by the Brauer–Wall group $\mathrm{BW}(k)$ and the elements represented by quadratic extensions $k\langle\sqrt{\alpha}\rangle$, where $0 \neq \alpha \in k$. In other words, division algebras with trivial H_4-action and quadratic extensions with a natural H_4-action represent the generators of the group $\mathrm{BM}(k, H_4, R_0)$. In the end of the section, we apply our structure theorems to Azumaya algebras over the triangular Hopf algebra $E(n)$ studied in [16]. We assign to each $(E(n), R_0)$-Azumaya algebra a $n \times n$-symmetric matrix invariant. Thus [2, Theorem 4.4] will follow from our structure theorems.

In Section 4, we deal with H_4-Azumaya algebras with respect to the triangular structures R_t, where $t \neq 0$. It is well known that the triangular Hopf algebra (H_4, R_t) is a cocycle twist of (H_4, R_0) for any $t \in k$ (e.g. see [5]) and consequently the two braided monoidal categories ${}_{H_4}\mathcal{M}^{R_0}$ and ${}_{H_4}\mathcal{M}^{R_t}$ are equivalent. It follows from [20] that the Brauer groups $\mathrm{BM}(k, H_4, R_0)$ and $\mathrm{BM}(k, H_4, R_t)$ are isomorphic. We consider the structures of H_4-Azumaya algebras in ${}_{H_4}\mathcal{M}^{R_0}$ under cocycle twisting. We then obtain the structure theorems of H_4-Azumaya algebras in the category ${}_{H_4}\mathcal{M}^{R_t}$ (see Theorems 4.4 and 4.5). Using the structure theorems we can assign to each Azumaya algebra in ${}_{H_4}\mathcal{M}^{R_t}$ a scalar invariant in a less natural manner.

1. Preliminaries

1.1. Yetter–Drinfeld module algebras

Throughout, we work over a fixed field k with $\mathrm{ch}(k) \neq 2$. Unless otherwise stated, all algebras, Hopf algebras and modules are defined over k; all maps are k-linear; dim, $\otimes$ and Hom stand for $\dim_k$, $\otimes_k$ and Hom_k, respectively. For the theory of Hopf algebras and quantum groups, we refer to [9,13,15,18]. We denote by k^+ the additive group of k, by $k^\times$ the multiplicative group of all nonzero elements of k, and by $k^{\times 2}$ the subgroup $\{\alpha^2 \mid \alpha \in k^\times\}$ of nonzero square elements.

Let H be a Hopf algebra with a bijective antipode S. A Yetter–Drinfeld H-module (simply, YD H-module) M is a crossed H-bimodule. That is, M is simultaneously a left H-module and a right H-comodule satisfying the following equivalent compatibility conditions:

(i) $\sum h_{(1)} \cdot m_{(0)} \otimes h_{(2)} m_{(1)} = \sum (h_{(2)} \cdot m)_{(0)} \otimes (h_{(2)} \cdot m)_{(1)} h_{(1)}$,
(ii) $\sum (h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = \sum h_{(2)} \cdot m_{(0)} \otimes h_{(3)} m_{(1)} S^{-1}(h_{(1)})$,

where $h \in H$, $m \in M$, and the sigma notations for a comodule and for a comultiplication can be found in the reference book [18]. A Yetter–Drinfeld H-module algebra (simply, YD H-module algebra) is a YD H-module A such that A is a left H-module algebra and a right H^{op}-comodule algebra. For the detail of H-(co)module algebras we refer to [9,13, 15,18].

Let A and B be two YD H-module algebras. We may define a *braided product*, denoted #, on the YD H-module $A \otimes B$:

$$(a \# b)(c \# d) = \sum ac_{(0)} \# (c_{(1)} \cdot b)d$$

for $a, c \in A$ and $b, d \in B$. The braided product # makes $A \# B$ a left H-module algebra and a right H^{op}-comodule algebra so that $A \# B$ is a YD H-module algebra.

Now let A be a YD H-module algebra. The *H-opposite algebra* $\overline{A}$ of A is the YD H-module algebra defined as follows: $\overline{A}$ equals A as a YD H-module, but with multiplication given by the formula

$$\bar{a} \circ \bar{b} = \sum \overline{b_{(0)}(b_{(1)} \cdot a)}$$

for all $\bar{a}, \bar{b} \in \overline{A}$.

Let M be a finite-dimensional YD H-module. Then the dual space M^* is also a YD H-module with the H-module and H-comodule structures given by

(i) $(h \cdot m^*)(m) = m^*(S(h) \cdot m)$,
(ii) $\sum \langle m^*_{(0)}, m\rangle m^*_{(1)} = \sum \langle m^*, m_{(0)}\rangle S^{-1}(m_{(1)})$,

where $h \in H$, $m^* \in M^*$ and $m \in M$. The endomorphism algebra $\mathrm{End}_k(M)$ is a YD H-module algebra with the H-structures induced by those of M, i.e., for $h \in H$, $f \in \mathrm{End}_k(M)$ and $m \in M$,

(i) $(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m)$,
(ii) $\chi(f)(m) = \sum f(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)})_{(1)}$.

Recall from [3, 4.2] that the H-opposite of $\mathrm{End}_k(M)$ is isomorphic to $\mathrm{End}_k(M)^{\mathrm{op}}$ as a YD H-module algebra, where the latter has a slightly different YD H-module structure (see [3] for the details).

A YD H-module algebra is called *elementary* if it is isomorphic to $\mathrm{End}(M)$ for some finite-dimensional YD H-module M.

Now let H be a quasitriangular Hopf algebra, that is, H is a Hopf algebra with an invertible element $R = \sum R^{(1)} \otimes R^{(2)}$ in $H \otimes H$ satisfying the following axioms ($r = R$):

(QT1) $\sum \Delta(R^{(1)}) \otimes R^{(2)} = \sum R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$,
(QT2) $\sum \varepsilon(R^{(1)}) R^{(2)} = 1$,
(QT3) $\sum R^{(1)} \otimes \Delta(R^{(2)}) = \sum R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$,
(QT4) $\sum R^{(1)} \varepsilon(R^{(2)}) = 1$,
(QT5) $\Delta^{\mathrm{cop}}(h) R = R \Delta(h)$, $\forall h \in H$,

where $\Delta^{\mathrm{cop}} = \tau \Delta$ is the comultiplication of the Hopf algebra H^{cop} and τ is the flip map. R is called a quasitriangular structure. If, in addition, $R^{-1} = \tau(R)$, then R is called a triangular structure. In this case, we also say that H is a triangular Hopf algebra.

If A is a left H-module (algebra), then A is simultaneously a YD H-module (algebra) with the right H^{op}-comodule structure given by

$$A \to A \otimes H^{\mathrm{op}}, \qquad a \mapsto \sum \big(R^{(2)} \cdot a\big) \otimes R^{(1)}, \quad a \in A. \tag{1}$$

1.2. H-Azumaya algebras

A YD H-module algebra A is called H-Azumaya if the following YD H-module algebra maps are isomorphisms:

$$F : A \# \overline{A} \to \mathrm{End}(A), \qquad F(a \# \bar{b})(c) = \sum a c_{(0)}(c_{(1)} \cdot b),$$
$$G : \overline{A} \# A \to \mathrm{End}(A)^{\mathrm{op}}, \qquad G(\bar{a} \# b)(c) = \sum a_{(0)}(a_{(1)} \cdot c)b.$$

If a YD H-module algebra is H-Azumaya, then its right center

$$Z^r(A) := \Big\{ x \in A \;\Big|\; xy = \sum y_{(0)}(y_{(1)} \cdot x),\ \forall y \in A \Big\},$$

and its left center

$$Z^l(A) := \Big\{ x \in A \;\Big|\; yx = \sum x_{(0)}(x_{(1)} \cdot y),\ \forall y \in A \Big\}$$

are both trivial (see [4]).

Let H be a quasitriangular Hopf algebra with a quasitriangular structure $R = \sum R^{(1)} \otimes R^{(2)}$. Then the category ${}_H\mathcal{M}$ of the left H-modules is a braided category, denoted by ${}_H\mathcal{M}^R$. Explicitly, let M and N be left H-modules. Then there is an H-module isomorphism

$$\Phi^R_{M,N} : M \otimes N \to N \otimes M, \qquad m \otimes n \mapsto \sum R^{(2)} \cdot n \otimes R^{(1)} \cdot m.$$

In fact, ${}_H\mathcal{M}^R$ is a braided monoidal full subcategory of $\mathcal{Q}^H$ (or ${}_H\mathcal{YD}^H$), the category of Yetter–Drinfeld H-modules, via (1). In this case, a left H-module algebra A is called *(H, R)-Azumaya* if A is H-Azumaya as a YD H-module algebra with the right H^{op}-comodule structure (1). If A and B are two left H-module algebras, then we denote the braided product $A \# B$ in ${}_H\mathcal{M}^R \subset \mathcal{Q}^H$ by $A \#_R B$.

Let A be a YD H-module algebra and let B be a YD H-module subalgebra of A. Define the left and right H-centralizers of B in A by

$$C^l_A(B) = \Big\{ x \in A \;\Big|\; yx = \sum x_{(0)}(x_{(1)} \cdot y),\ \forall y \in B \Big\}$$

and

$$C^r_A(B) = \Big\{ x \in A \;\Big|\; xy = \sum y_{(0)}(y_{(1)} \cdot x),\ \forall y \in B \Big\},$$

respectively. When $B = A$, we have $Z^r(A) = C^r_A(A)$ and $Z^l(A) = C^l_A(A)$.

Now assume that R is a quasitriangular structure of H. Let A be a left H-module algebra and let B be an H-module subalgebra of A. Then A is a Yetter–Drinfeld H-module algebra and B is a Yetter–Drinfeld H-module subalgebra of A with the H^{op}-comodule structure (1) given by R. In this case, we denote $C_A^l(B)$ and $C_A^r(B)$ by ${}^RC_A^l(B)$ and ${}^RC_A^r(B)$, respectively. If R is triangular, then the braiding Φ^R of ${}_H\mathcal{M}^R$ is symmetric, i.e., $\Phi_{N,M}^R\Phi_{M,N}^R = \mathrm{id}_{M\otimes N}$ for any object M and N in ${}_H\mathcal{M}^R$. In this case, ${}^RC_A^l(B) = {}^RC_A^r(B)$, denoted by $\widehat{C}_A^R(B)$.

1.3. $\mathbb{Z}_2$-graded Azumaya algebras

Let k be a field. In [22], Wall introduced the notion of a *$\mathbb{Z}_2$-graded Azumaya algebra*, which is a graded central and a graded simple algebra (CSGA) A in the following sense:

(i) the graded center $Z_g(A) = \{a \in A \mid ab = ba_0 + b_0a_1 - b_1a_1, \forall b \in A\} = k$.
(ii) A is a simple graded algebra, i.e., A has no proper nonzero graded ideals.

If the characteristic of k is different from 2, we may replace 'graded simplicity' (ii) by 'graded separability,' that is,

(iii) The multiplication map $m : A \otimes A \to A$ splits as a $\mathbb{Z}_2$-graded A-bimodule map, where the grading on $A \otimes A$ is the diagonal one.

Given two graded algebras A and B. The graded product $A \# B$ of A and B is defined as follows:

$$(a \# b)(c \# d) = (-1)^{\partial(b)\partial(c)} ac \# bd, \tag{2}$$

where b and c are homogeneous elements and $\partial(b)$, $\partial(c)$ are the graded degrees of b and c, respectively. If A and B are graded Azumaya algebras, then the graded product $A \# B$ is a graded Azumaya algebra. Now one may repeat the definition of $\mathrm{Br}(k)$ by adding the term '($\mathbb{Z}_2$-) graded' to obtain the Brauer group of graded algebras which is referred to as *the Brauer–Wall group*, denoted $\mathrm{BW}(k)$. The Brauer–Wall group is an example of the equivariant Brauer group of a quasitriangular Hopf algebra. Let $k\mathbb{Z}_2$ be the group Hopf algebra. The Hopf algebra $k\mathbb{Z}_2$ has a triangular structure

$$R_0 = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g),$$

where g is the nonidentity of $\mathbb{Z}_2$. An algebra A is a $\mathbb{Z}_2$-graded algebra if and only if it is a left $k\mathbb{Z}_2$-module algebra if and only if it has an involution. Thus, the braided product of the category ${}_{k\mathbb{Z}_2}\mathcal{M}^{R_0}$ induced by the above triangular structure R_0 is exactly the graded product (2). Thus a $\mathbb{Z}_2$-graded algebra is a CSGA if and only if it is a $(k\mathbb{Z}_2, R_0)$-Azumaya algebra. So the Brauer–Wall group $\mathrm{BW}(k)$ is the equivariant Brauer group $\mathrm{BM}(k, k\mathbb{Z}_2, R_0)$ (see [21] for more detail).

The Brauer–Wall group $\mathrm{BW}(k)$ can be completely described in terms of the usual Brauer group $\mathrm{Br}(k)$ and the group of graded quadratic extensions:

$$1 \to \mathrm{Br}(k) \to \mathrm{BW}(k) \to Q_2(k) \to 1,$$

where $Q_2(k) = \mathbb{Z}_2 \times (k^\times/k^{\times 2})$ with multiplication given by

$$(e,d)\big(e',d'\big) = \big(e+e', (-1)^{ee'}dd'\big)$$

for $e, e' \in \mathbb{Z}_2$ and $d, d' \in k^\times/k^{\times 2}$, where we let $\mathbb{Z}_2 = \{0,1\}$. One may write down the multiplication rule for the product $\mathrm{Br}(k) \times Q_2(k)$ to see that $\mathrm{BW}(k)$ is isomorphic to $\mathrm{Br}(k) \times Q_2(k)$ (for the detail see [8,11,17]).

1.4. Hopf algebras twisted by cocycles

Let H be a Hopf algebra over k. Recall from [13] that an invertible element $\theta = \sum \theta^{(1)} \otimes \theta^{(2)} \in H \otimes H$ is called a *dual* (*left*) 2-*cocycle* (also called a Drinfeld twist in the literature) if it satisfies:

$$(\theta \otimes 1)(\Delta \otimes \mathrm{id})(\theta) = (1 \otimes \theta)(\mathrm{id} \otimes \Delta)(\theta).$$

and $(\varepsilon \otimes \mathrm{id})(\theta) = (\mathrm{id} \otimes \varepsilon)(\theta) = 1$.

To a dual 2-cocycle $\theta \in H \otimes H$ one may associate a new Hopf algebra H^θ. As an algebra $H^\theta = H$ but with comultiplication given by

$$\Delta^\theta(h) = \theta\Delta(h)\theta^{-1}, \quad h \in H_\theta.$$

The antipode S^θ of H^θ is given by

$$\begin{aligned} S^\theta(h) &= \sum \theta^{(1)} S\big(\theta^{(2)}\big) S(h) S\big(\big(\theta^{-1}\big)^{(1)}\big)\big(\theta^{-1}\big)^{(2)} \\ &= \sum \theta^{(1)} S\big(\big(\theta^{-1}\big)^{(1)} h\theta^{(2)}\big)\big(\theta^{-1}\big)^{(2)}, \quad h \in H^\theta, \end{aligned}$$

where $\theta = \sum \theta^{(1)} \otimes \theta^{(2)}$ and $\theta^{-1} = \sum (\theta^{-1})^{(1)} \otimes (\theta^{-1})^{(2)}$ in $H \otimes H$. Note that S^θ is bijective since S is bijective.

A dual 2-cocycle $\theta \in H \otimes H$ is *lazy* if $\theta\Delta(h) = \Delta(h)\theta$ for all $h \in H$. This is equivalent to $\Delta^\theta = \Delta$, i.e., $H^\theta = H$ as Hopf algebras.

Now assume that $\theta = \sum \theta^{(1)} \otimes \theta^{(2)} \in H \otimes H$ is a dual 2-cocycle. Let $R = \sum R^{(1)} \otimes R^{(2)}$ be a quasitriangular structure of H. Put

$$R^\theta := \tau(\theta) R \theta^{-1} = \sum \theta^{(2)} R^{(1)} \big(\theta^{-1}\big)^{(1)} \otimes \theta^{(1)} R^{(2)} \big(\theta^{-1}\big)^{(2)}.$$

Then R^θ is a quasitriangular structure of H^θ [13, Theorem 2.3.4]. It is well known that θ induces a braided monoidal category equivalence functor (Θ, ϕ) from ${}_H\mathcal{M}^R$ to ${}_{H^\theta}\mathcal{M}^{R^\theta}$

(e.g., see [7]). The functor (Θ, ϕ) can be described as follows. Since $H^\theta = H$ as algebras, an H^θ-module is the same as an H-module. For any object M and a morphism f in ${}_H\mathcal{M}^R$, let $\Theta(M) = M$ and $\Theta(f) = f$ in ${}_{H^\theta}\mathcal{M}^{R^\theta}$. That is, Θ is the identity functor from ${}_H\mathcal{M}$ to ${}_{H^\theta}\mathcal{M} = {}_H\mathcal{M}$ as categories without monoidal structures. For any objects M and N in ${}_H\mathcal{M}^R$, let

$$\phi_{M,N} : \Theta(M) \otimes \Theta(N) \to \Theta(M \otimes N)$$
$$m \otimes n \mapsto \theta^{-1} \cdot (m \otimes n) = \sum \left(\theta^{-1}\right)^{(1)} \cdot m \otimes \left(\theta^{-1}\right)^{(2)} \cdot n.$$

Then $\phi_{M,N}$ is a family of natural isomorphisms in ${}_{H^\theta}\mathcal{M}^{R^\theta}$. The functor (Θ, ϕ) is a braided monoidal category equivalence functor from ${}_H\mathcal{M}^R$ to ${}_{H^\theta}\mathcal{M}^{R^\theta}$ (see [7]). In particular, if θ is lazy, then (Θ, ϕ) is a braided monoidal category equivalence functor from ${}_H\mathcal{M}^R$ to ${}_H\mathcal{M}^{R^\theta}$.

Let A be a left H-module algebra with multiplication $m : A \otimes A \to A$. Then $\Theta(A)$ is a left H^θ-module algebra with multiplication given by

$$m^\theta : \Theta(A) \otimes \Theta(A) \xrightarrow{\phi_{A,A}} \Theta(A \otimes A) \xrightarrow{\Theta(m)} \Theta(A).$$

Denote by $a \bullet b$ the product of elements a and b in $\Theta(A)$. Then

$$a \bullet b = \sum \left(\left(\theta^{-1}\right)^{(1)} \cdot a\right)\left(\left(\theta^{-1}\right)^{(2)} \cdot b\right), \quad a, b \in \Theta(A).$$

The following lemma now follows from [19] and the foregoing discussion.

Lemma 1.1. *Let (H, R) be a quasitriangular Hopf algebra, and $\theta \in H \otimes H$ a lazy dual 2-cocycle in H. Let A and B be two left H-module algebras. The following statements hold.*

(a) *A is (H, R)-Azumaya if and only if $\Theta(A)$ is (H, R^θ)-Azumaya.*
(b) *$\Theta(A) \#_{R^\theta} \Theta(B) \cong \Theta(A \#_R B)$ as left H-module algebras.*
(c) *If B is an H-module subalgebra of A, then ${}^{R^\theta}C^l_{\Theta(A)}(\Theta(B)) = {}^RC^l_A(B)$ and ${}^{R^\theta}C^r_{\Theta(A)}(\Theta(B)) = {}^RC^r_A(B)$ as k-spaces. Consequently, $\Theta({}^RC^l_A(B)) \cong {}^{R^\theta}C^l_{\Theta(A)}(\Theta(B))$ and $\Theta({}^RC^r_A(B)) \cong {}^{R^\theta}C^r_{\Theta(A)}(\Theta(B))$ as left H-module algebras.*

1.5. Sweedler's 4-dimensional Hopf algebra H_4

Sweedler's 4-dimensional Hopf algebra H_4 is generated by two elements g and h subject to the relations:

$$g^2 = 1, \qquad h^2 = 0, \qquad gh + hg = 0.$$

The coalgebra structure and the antipode are determined by

$$\Delta(g) = g \otimes g, \qquad \Delta(h) = h \otimes g + 1 \otimes h, \qquad \varepsilon(g) = 1,$$
$$\varepsilon(h) = 0, \qquad S(g) = g^{-1} = g, \qquad S(h) = gh.$$

Moreover, H_4 has a canonical basis $\{1, g, h, gh\}$. It is well known that H_4 is a triangular Hopf algebra with a family of triangular structures indexed by parameter $t \in k$:

$$\begin{aligned} R_t = {} & \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) \\ & + \frac{t}{2}(1 \otimes 1 + g \otimes g + 1 \otimes g - g \otimes 1)(h \otimes h). \end{aligned}$$

A left H_4-module algebra A is also called a *differential super algebra* (DSA) (see [21]). For A has a natural grading with $A_i = \{a \in A \mid g \cdot a = (-1)^i a\}$, $i = 0, 1$, and has a graded differential δ given by the action of $h \in H_4$, so that $\delta^2 = 0$ and

$$\delta(ab) = a\delta(b) + (-1)^i \delta(a)b$$

for all elements $a \in A$ and $b \in A_i$, $i = 0, 1$. If A and B are two left H_4-module algebras, then the braided product $A \#_{R_0} B$ is simply the graded product (2).

Conversely, let A be a $\mathbb{Z}_2$-graded algebra. Then A is an H_4-module algebra with the H_4-module structure given by

$$g \cdot x = (-1)^i x \quad \text{and} \quad h \cdot x = 0, \quad x \in A_i,\ i = 0, 1. \tag{3}$$

Denote by $\lfloor A \rfloor$ the H_4-module algebra with the derived H_4-action (3) on A.

Now let A be a left H_4-module algebra. The left and right centers $Z^l(A)$ and $Z^r(A)$ with respect to the triangular structure R_0 are exactly the graded center of A. Thus, an H_4-module algebra A is (H_4, R_0)-Azumaya if and only if A is $(k\mathbb{Z}_2, R_0)$-Azumaya, i.e., A is a CSGA.

Let $\alpha, \beta \in k$. Let $k(\sqrt{\alpha}\,)$ denote the k-algebra generated by one generator x subject to the relation $x^2 = \alpha$. Then $k(\sqrt{\alpha}\,)$ is an H_4-module algebra with the H_4-action given by

$$g \cdot x = -x, \qquad h \cdot x = \beta. \tag{4}$$

Denote by $k\langle\sqrt{\alpha}; \beta\rangle$ the H_4-module algebra $k(\sqrt{\alpha}\,)$ with the H_4-module structure given by (4).

By the foregoing discussion we know that $k\langle\sqrt{\alpha}; \beta\rangle$ is also a $\mathbb{Z}_2$-graded algebra with $k\langle\sqrt{\alpha}; \beta\rangle_0 = k1$ and $k\langle\sqrt{\alpha}; \beta\rangle_1 = kx$. When $\beta = 2\alpha$, we write $k\langle\sqrt{\alpha}\,\rangle$ for $k\langle\sqrt{\alpha}; 2\alpha\rangle$ for convenience. It is easy to see that $k\langle\sqrt{\alpha}; \beta\rangle$ is a CSGA for any $\alpha \in k^\times$. So we have the following.

Lemma 1.2.

(a) *$k\langle\sqrt{\alpha}; \beta\rangle$ is an (H_4, R_0)-Azumaya algebra for any $\alpha \in k^\times$ and $\beta \in k$.*

(b) *For $\alpha, \beta, \alpha', \beta' \in k$, $k\langle\sqrt{\alpha}; \beta\rangle \cong k\langle\sqrt{\alpha'}; \beta'\rangle$ as H_4-module algebras if and only if $\alpha = \delta^2\alpha'$ and $\beta = \delta\beta'$ for some $\delta \in k^\times$.*

It is natural to ask when the H_4-module algebra $k\langle\sqrt{\alpha}; \beta\rangle$ is (H_4, R_t)-Azumaya if $t \in k$ is not zero. To answer this question, we recall the braided monoidal equivalence from ${}_{H_4}\mathcal{M}^{R_0}$ to ${}_{H_4}\mathcal{M}^{R_t}$ for any $t \in k$. The equivalence is given by a braided monoidal functor Θ induced by a lazy dual cocycle θ in H_4.

It is not difficult to compute that H_4 has a family of lazy dual 2-cocycles parameterized by elements $s \in k$:

$$\theta_s = 1 \otimes 1 + \frac{s}{2}(h \otimes gh).$$

For any $s, t \in k$, we have that

$$\theta_s\theta_t = \theta_{s+t} \quad \text{and} \quad \theta_s^{-1} = \theta_{-s}.$$

The family of triangular structures R_t can be obtained by cocycle twisting one particular triangular structure, say R_0. In fact, we have the following twisting identity:

$$R_t^{\theta_s} = \tau(\theta_s)R_t\theta_s^{-1} = \tau(\theta_s)R_t\theta_{-s} = R_{t-s}.$$

It follows from Section 1.4 that θ_t induces a braided monoidal equivalence functor Θ_t from ${}_{H_4}\mathcal{M}^{R_t}$ to ${}_{H_4}\mathcal{M}^{R_0}$ for any $t \in k$. Now let A be a left H_4-module algebra, and $t \in k$. The H_4-module algebra $\Theta_t(A)$ has the multiplication given by

$$a \bullet b = \sum \big(\theta_{-t}^{(1)} \cdot a\big)\big(\theta_{-t}^{(2)} \cdot b\big) = ab - \frac{t}{2}(h \cdot a)(gh \cdot b) \tag{5}$$

for all a and $b \in \Theta_t(A)$.

We end this section by giving a necessary and sufficient condition for $k\langle\sqrt{\alpha}; \beta\rangle$ to be (H_4, R_t)-Azumaya, for any $t \in k$.

Lemma 1.3. *For $\alpha, \beta, \alpha', \beta'$ and t in k, we have*

(a) *$\Theta_t(k\langle\sqrt{\alpha}; \beta\rangle) \cong k\langle\sqrt{\alpha - \frac{t}{2}\beta^2}; \beta\rangle$ as H_4-module algebras.*
(b) *The algebra $k\langle\sqrt{\alpha}; \beta\rangle$ is (H_4, R_t)-Azumaya if and only if $2\alpha - t\beta^2 \neq 0$.*
(c) *$k\langle\sqrt{\alpha}; \beta\rangle \#_{R_t} k\langle\sqrt{\alpha'}; \beta'\rangle \cong (\frac{\alpha, \alpha', t\beta\beta'}{k})$ as algebras, where the latter is the generalized quaternion algebra.*

Proof. (a) Let x be the generator of $k\langle\sqrt{\alpha}; \beta\rangle$ such that $x^2 = \alpha$, $g \cdot x = -x$ and $h \cdot x = \beta$. By definition, we have $x \bullet x = x^2 - \frac{t}{2}(h \cdot x)(gh \cdot x) = \alpha - \frac{t}{2}\beta^2$ in $\Theta_t(k\langle\sqrt{\alpha}; \beta\rangle)$. It follows that $\Theta_t(k\langle\sqrt{\alpha}; \beta\rangle) \cong k\langle\sqrt{\alpha - \frac{t}{2}\beta^2}; \beta\rangle$ as H_4-module algebras.

(b) By part (a), Lemma 1.1(a) and Lemma 1.2, we have that $k\langle\sqrt{\alpha};\beta\rangle$ is (H_4, R_t)-Azumaya if and only if $\Theta_t(k\langle\sqrt{\alpha};\beta\rangle)\cong k\langle\sqrt{\alpha-\frac{t}{2}\beta^2};\beta\rangle$ is (H_4, R_0)-Azumaya if and only if $\alpha-\frac{t}{2}\beta^2\neq 0$.

(c) Observe that $k\langle\sqrt{\alpha};\beta\rangle \#_{R_t} k\langle\sqrt{\alpha'};\beta'\rangle$ is generated, as an algebra, by $x\#1$ and $1\#x'$. Now by a straightforward computation, one gets that $(x\#1)^2=\alpha$, $(1\#x')^2=\alpha'$ and $(x\#1)(1\#x')+(1\#x')(x\#1)=t\beta\beta'$. It follows that $k\langle\sqrt{\alpha};\beta\rangle \#_{R_t} k\langle\sqrt{\alpha'};\beta'\rangle\cong(\frac{\alpha,\alpha',t\beta\beta'}{k})$ as algebras. □

2. The centralizer theorem

In this section, we generalize the classical centralizer theorem of Azumaya algebras to the centralizer theorem of H-Azumaya algebras, which will be used to obtain our structure theorems in the next two sections. We will let H be a Hopf algebra with a bijective antipode in this section. We begin by considering when the product of two YD H-module algebras is H-Azumaya.

Lemma 2.1. *Let A and B be two* YD *H-module algebras. Then $A\#B$ is H-Azumaya if and only if both A and B are H-Azumaya.*

Proof. It is enough to show the necessity part. The sufficiency part is clear. Suppose that $A\#B$ is H-Azumaya. Consider the composition of YD H-module algebra homomorphisms

$$\begin{aligned}
(A\#B)\#(\overline{A\#B}) &\cong (A\#B)\#(\overline{B}\#\overline{A})\\
&\cong A\#\big((B\#\overline{B})\#\overline{A}\big)\\
&\xrightarrow{\mathrm{id}\otimes(F_B\otimes\mathrm{id})} A\#\big(\mathrm{End}(B)\#\overline{A}\big)\\
&\cong A\#\big(\overline{A}\#\mathrm{End}(B)\big)\\
&\cong (A\#\overline{A})\#\mathrm{End}(B)\\
&\xrightarrow{F_A\otimes\mathrm{id}} \mathrm{End}(A)\#\mathrm{End}(B)\\
&\cong \mathrm{End}(A\#B),
\end{aligned}$$

where we use [3, Proposition 2.4.4(c)] in the first isomorphism, the second and fifth isomorphisms follow from [3, Proposition 2.4.2], the fourth isomorphism follows from [3, Proposition 4.6], and we use [3, Proposition 4.3] in the last isomorphism. Since $A\#B$ is H-Azumaya, $(A\#B)\#(\overline{A\#B})\cong\mathrm{End}(A\#B)$ as YD H-module algebras. Hence $(A\#B)\#(\overline{A\#B})$ is a simple algebra. It follows that any algebra homomorphism from $(A\#B)\#(\overline{A\#B})$ to $\mathrm{End}(A\#B)$ is an injection, and so it is an isomorphism by comparing their dimensions. Therefore, the composition given above is an isomorphism. Thus $\mathrm{id}\otimes(F_B\otimes \mathrm{id}): A\ \#\ ((B\ \#\ \overline{B})\ \#\ \overline{A}) \to A\ \#\ (\mathrm{End}(B)\ \#\ \overline{A})$ is injective, and $F_A\otimes \mathrm{id}: (A\ \#\ \overline{A})\ \#\ \mathrm{End}(B) \to \mathrm{End}(A)\ \#\ \mathrm{End}(B)$ is surjective. Hence F_B is a monomorphism and F_A is an epimorphism, and so both F_A and F_B are isomorphisms by comparing the dimensions of the corresponding YD H-module algebras. Similarly, one can show that both $G_A: \overline{A}\ \#\ A \to \mathrm{End}(A)^{\mathrm{op}}$ and $G_B: \overline{B}\ \#\ B \to \mathrm{End}(B)^{\mathrm{op}}$ are isomorphisms. Hence both A and B are H-Azumaya. □

Next we consider the left and right centralizers of a YD H-module subalgebra B in a YD H-module algebra A. We show that both centralizers are YD H-module subalgebras.

Lemma 2.2. *Let A be a* YD *H-module algebra. Let B be a* YD *H-module subalgebra of A. Then both $C_A^l(B)$ and $C_A^r(B)$ are* YD *H-module subalgebras of A.*

Proof. It is easy to see that $C_A^l(B)$ is a k-subspace of A and contains the identity 1 of A. Let $x, x' \in C_A^l(B)$, $y \in B$ and $h \in H$. Since B is an H-submodule of A, we have

$$yxx' = \sum x_{(0)}(x_{(1)}\cdot y)x' = \sum x_{(0)}x'_{(0)}\big(x'_{(1)}x_{(1)}\cdot y\big) = \sum \big(xx'\big)_{(0)}\big(\big(xx'\big)_{(1)}\cdot y\big)$$

and

$$\begin{aligned} y(h\cdot x) &= \sum h_{(2)}\cdot\big(\big(S^{-1}(h_{(1)})\cdot y\big)x\big) = \sum h_{(2)}\cdot\big(x_{(0)}\big(x_{(1)}S^{-1}(h_{(1)})\cdot y\big)\big) \\ &= \sum (h_{(2)}\cdot x_{(0)})\big(h_{(3)}x_{(1)}S^{-1}(h_{(1)})\cdot y\big) \\ &= \sum (h\cdot x)_{(0)}\big((h\cdot x)_{(1)}\cdot y\big). \end{aligned}$$

This shows that $C_A^l(B)$ is a subalgebra and an H-submodule of A. Since B is an H-subcomodule of A, we also have

$$\begin{aligned} \sum yx_{(0)}\otimes x_{(1)} &= \sum y_{(0)}x_{(0)}\otimes x_{(1)}y_{(1)}S(y_{(2)}) \\ &= \sum (y_{(0)}x)_{(0)}\otimes (y_{(0)}x)_{(1)}S(y_{(1)}) \\ &= \sum \big(x_{(0)}(x_{(1)}\cdot y_{(0)})\big)_{(0)}\otimes \big(x_{(0)}(x_{(1)}\cdot y_{(0)})\big)_{(1)}S(y_{(1)}) \\ &= \sum x_{(0)}(x_{(2)}\cdot y_{(0)})_{(0)}\otimes (x_{(2)}\cdot y_{(0)})_{(1)}x_{(1)}S(y_{(1)}) \\ &= \sum x_{(0)}(x_{(1)}\cdot y_{(0)})\otimes x_{(2)}y_{(1)}S(y_{(2)}) \\ &= \sum x_{(0)}(x_{(1)}\cdot y)\otimes x_{(2)} \\ &= \sum x_{(0)(0)}(x_{(0)(1)}\cdot y)\otimes x_{(1)}. \end{aligned}$$

It follows that $C_A^l(B)$ is a H-subcomodule of A. Hence $C_A^l(B)$ is a YD H-module subalgebra of A. Similarly, one can check that $C_A^r(B)$ is a YD H-module subalgebra of A. □

Recall from [4] that an H-Azumaya algebra B defines a Morita equivalence between the category $\mathcal{Q}^H$ of YD H-modules and the category ${}_{B\#\overline{B}}\mathcal{Q}^H$ of $B \# \overline{B}$-modules in the category $\mathcal{Q}^H$:

$$B \otimes - : \mathcal{Q}^H \to {}_{B\#\overline{B}}\, \mathcal{Q}^H, \qquad (-)^B : {}_{B\#\overline{B}}\mathcal{Q}^H \to \mathcal{Q}^H$$

where $M^B = \{m \in M \mid (1 \# \bar{b})m = (b \# 1)m,\ \forall b \in B\}$, for $M \in_{B\#\overline{B}} \mathcal{Q}^H$. Now if A is a YD H-module algebra and B is a YD H-module subalgebra such that B is H-Azumaya. Then A has a natural left $B \# \overline{B}$-module structure given by

$$(c \# \bar{b})a = \sum ca_{(0)}(a_{(1)} \cdot b), \quad b, c \in B,\ a \in A.$$

In this case, we have $A^B = \{a \in A \mid ba = \sum a_{(0)}(a_{(1)} \cdot b),\ \forall b \in B\}$, which is exactly the left centralizer $C_A^l(B)$. By [4, Proposition 2.6] we have a YD H-module isomorphism

$$B \otimes C_A^l(B) \to A, \qquad b \otimes c \mapsto bc. \tag{6}$$

On the other hand, the H-Azumaya algebra B defines an equivalence between the category $\mathcal{Q}^H$ and the category $\mathcal{Q}^H_{\overline{B}\#B}$ of right $\overline{B} \# B$-modules in $\mathcal{Q}^H$:

$$- \otimes B : \mathcal{Q}^H \to \mathcal{Q}^H_{\overline{B}\#B}, \qquad {}^B(-) : \mathcal{Q}^H_{\overline{B}\#B} \to \mathcal{Q}^H$$

where ${}^BM = \{m \in M \mid m(\bar{b} \# 1) = m(1 \# b),\ \forall b \in B\}$ for $M \in \mathcal{Q}^H_{\overline{B}\#B}$. We have a natural $\overline{B} \# B$-action on A as follows:

$$a(\bar{b} \# c) = \sum b_{(0)}(b_{(1)} \cdot a)c, \quad \forall b, c \in B,\ a \in A.$$

The YD H-submodule ${}^BA = \{a \in A \mid a(\bar{b} \# 1) = a(1 \# b), \forall b \in B\}$ coincides with the right centralizer $C_A^r(B)$. Thus, we obtain a YD H-module isomorphism:

$$C_A^r(B) \otimes B \to A, \qquad c \otimes b \mapsto cb. \tag{7}$$

Now we are ready to give the centralizer theorem of H-Azumaya algebras.

Theorem 2.3. *Let A be a* YD *H-module algebra, and let B be a* YD *H-module subalgebra of A. Assume that A and B are H-Azumaya. Then both $C_A^l(B)$ and $C_A^r(B)$ are H-Azumaya. Moreover, we have $C_A^l(B) \# B \cong A$, $B \# C_A^r(B) \cong A$ as* YD *H-module algebras, $C_A^r(C_A^l(B)) = B$ and $C_A^l(C_A^r(B)) = B$.*

Proof. By the YD H-module isomorphisms $B \# C_A^l(B) \to A,\ b \otimes c \mapsto bc$ and $C_A^r(B) \# B \to A,\ c \otimes b \mapsto cb$ given in (6) and (7) respectively, we know that $A = BC_A^l(B) = C_A^r(B)B$ and $\dim A = (\dim B)(\dim C_A^l(B)) = (\dim B)(\dim C_A^r(B))$. Let $a \in C_A^l(B)$ and $b \in B$. Then by the definition of $C_A^l(B)$ and Lemma 2.2 one gets $ba = \sum a_{(0)}(a_{(1)} \cdot b) \in$

$C^l_A(B)B$. Hence $A = C^l_A(B)B$. Similarly, one can show that $A = BC^r_A(B)$. Thus there are two k-linear surjections

$$\varphi : C^l_A(B) \# B \to A, \qquad a \# b \mapsto ab$$

and

$$\psi : B \# C^r_A(B), \qquad b \# c \mapsto bc.$$

Since $\dim A = (\dim B)(\dim C^l_A(B)) = (\dim B)(\dim C^r_A(B))$, φ and ψ are k-linear isomorphisms. Since the multiplication map $m : A \otimes A \to A$ is a YD H-module map, so are φ and ψ. It remains to show that φ and ψ are algebra maps. This is clear because

$$\begin{aligned}
\varphi\big((a \# b)\big(a' \# b'\big)\big) &= \sum \varphi\big(aa'_{(0)} \# \big(a'_{(1)} \cdot b\big)b'\big) \\
&= \sum aa'_{(0)}\big(a'_{(1)} \cdot b\big)b' \\
&= aba'b' \\
&= \varphi(a \# b)\varphi\big(a' \# b'\big)
\end{aligned}$$

for $a, a' \in C^l_A(B)$ and $b, b' \in B$. Similarly, we have that ψ is an algebra isomorphism. Now since A is an H-Azumaya algebra, it follows from Lemma 2.1 that $C^l_A(B)$ and $C^r_A(B)$ are H-Azumaya algebras.

Furthermore, it is clear that $B \subseteq C^l_A(C^r_A(B))$ and $B \subseteq C^r_A(C^l_A(B))$. Replacing B with $C^r_A(B)$, we have $\dim A = (\dim C^r_A(B))(\dim C^l_A(C^r_A(B)))$. Hence $\dim B = \dim C^l_A(C^r_A(B))$ and so $B = C^l_A(C^r_A(B))$. Similarly, one can show $B = C^r_A(C^l_A(B))$. $\square$

Applying Theorem 2.3 to (H, R)-Azumaya algebras over a quasitriangular or a triangular Hopf algebra (H, R), we obtain the following corollary.

Corollary 2.4. *Assume that R is a quasitriangular structure of H. Let A be a left H-module algebra and let B be an H-module subalgebra of A. If both A and B are (H, R)-Azumaya, then so are ${}^R C^l_A(B)$ and ${}^R C^r_A(B)$. Moreover, we have H-module algebra isomorphisms*

$$A \cong {}^R C^l_A(B) \#_R B \cong B \#_R {}^R C^r_A(B).$$

If, in addition, R is triangular, then we have H-module algebra isomorphisms

$$A \cong \widehat{C}^R_A(B) \#_R B \cong B \#_R \widehat{C}^R_A(B).$$

Similarly, if we apply Theorem 2.3 to $(H, \mathcal{R})$-Azumaya algebras over a coquasitriangular Hopf algebra $(H, \mathcal{R})$, we may get a similar centralizer theorem to Corollary 2.4.

3. Structure theorems of (H_4, R_0)-Azumaya algebras

From now on, all algebras considered will be finite-dimensional over k. If A is an algebra, we denote by $Z(A)$ the center of A. Throughout this section, # will stand for the braided product $\#_{R_0}$, i.e., the graded product (2). For any subset B of A, we let $C_A(B)$ stand for the usual centralizer of B in A. That is,

$$C_A(B) = \{a \in A \mid ab = ba, \ \forall b \in B\},$$

a subalgebra of A. If A' is another subset of A, we let

$$C_{A'}(B) := A' \cap C_A(B) := \{a \in A' \mid ab = ba, \ \forall b \in B\}.$$

In case $B = \{x\}$, we simply write $C_{A'}(x)$ for $C_{A'}(\{x\})$.

Let A be an algebra. We can always endow A with a trivial $\mathbb{Z}_2$-grading. In order to avoid confusion, we will use $\underline{A}$, in stead of (A) in [11], to denote the trivial graded algebra A with

$$\underline{A}_0 = A \quad \text{and} \quad \underline{A}_1 = 0. \tag{8}$$

Now let $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded algebra, and let n be a positive integer. Then the matrix algebra $M_n(A)$ has a natural $\mathbb{Z}_2$-grading: $M_n(A)_i = M_n(A_i)$, $i \in \mathbb{Z}_2$. Denote this $\mathbb{Z}_2$-graded algebra by $\widetilde{M}_n(A)$. It is easy to see that $\widetilde{M}_n(k)$ is just $\underline{M_n(k)}$, and $\widetilde{M}_n(A) \cong \widetilde{M}_n(k) \# A = \widetilde{M}_n(k) \otimes A$.

Let A be an algebra, and let $u \in A$. If $u^2 = \delta \in k^\times$, then u induces a $\mathbb{Z}_2$-grading:

$$A_i = \{x \in A \mid uxu^{-1} = (-1)^i x\}, \quad i \in \mathbb{Z}_2. \tag{9}$$

Denote by A^u the $\mathbb{Z}_2$-graded algebra A with the grading induced by element u.

Lemma 3.1. *Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded algebra. For any $\alpha \in k^\times$, we let $A_{(\alpha)} = A$ as a graded vector space but with a new multiplication given by*

$$x \cdot_\alpha y = (x_0 + x_1) \cdot_\alpha (y_0 + y_1) = (x_0 y_0 + \alpha x_1 y_1) + (x_0 y_1 + x_1 y_0) \tag{10}$$

where $x = x_0 + x_1$, $y = y_0 + y_1 \in A$, and $x_i, y_i \in A_i$, $i \in \mathbb{Z}_2$. Then $A_{(\alpha)}$ is a graded algebra.

In fact, a $\mathbb{Z}_2$-graded algebra A can be regarded as a right $k\mathbb{Z}_2$-comodule algebra. A nonzero element α in k gives a 2-cocycle σ_α on $k\mathbb{Z}_2$; and $A_{(\alpha)}$ is just the σ_α-twist of A. Observe that $A_{(\alpha\delta^2)} \cong A_{(\alpha)}$ as $\mathbb{Z}_2$-graded algebras for any $\alpha, \delta \in k^\times$.

Now let A be a CSGA. The center $Z(A)$ of A is a graded algebra of the form $k \oplus Z_1$ with $Z_1 \subseteq A_1$. If $Z_1 \neq 0$, then we say that A is of *odd type*. If $Z_1 = 0$, then A is of *even type*. Assume $A_1 \neq 0$. By [11, Theorem IV.3.4], $Z_1 = 0$ if and only if A is a CSA over k (as an ungraded algebra); and $Z_1 \neq 0$ if and only if A_0 is a CSA over k. *An (H_4, R_0)-Azumaya algebra is said to be of even (or odd) type if it is of even (or odd) type as a* CSGA. The following two lemmas can be found in [11,22].

Lemma 3.2. [11, Theorem IV.3.6] *Let A be a* CSGA *of odd type. The following assertions hold.*

(a) $Z(A) = C_A(A_0) = k \oplus kz$, *where* $z \in Z_1$ *and* $z^2 = \alpha \in k^\times$. *The square class of* α *does not depend on the choice of* $z \in Z_1 \setminus \{0\}$, *and* $Z(A) \cong k\langle\sqrt{\alpha}\rangle$ *as graded algebras.*
(b) *There are graded algebra isomorphisms*

$$A \cong \underline{A_0} \# k\langle\sqrt{\alpha}\rangle = \underline{A_0} \otimes k\langle\sqrt{\alpha}\rangle,$$

where $\underline{A_0}$ *is the trivial graded algebra* A_0 *given in* (8).
(c) *If* $\alpha \notin k^{\times 2}$, *then* A *is a* CSA *over* $Z(A) \cong k(\sqrt{\alpha})$. *If* $\alpha \in k^{\times 2}$, *then* $Z(A) \cong k \times k$, *and* $A \cong A_0 \times A_0$.

In any case, A is a semisimple k-algebra.

Let A be an algebra. Let r and s be two nonnegative integers with $n = r + s > 0$. Then we can define a $\mathbb{Z}_2$-grading on the matrix algebra $M_n(A)$ by letting

$$M_n(A)_0 = \begin{pmatrix} M_r(A) & 0 \\ 0 & M_s(A) \end{pmatrix}, \qquad M_n(A)_1 = \begin{pmatrix} 0 & M_{r\times s}(A) \\ M_{s\times r}(A) & 0 \end{pmatrix}.$$

Denote by $M_n^{r+s}(A)$ the $\mathbb{Z}_2$-graded algebra. It is easy to see that $M_n^{r+s}(A)$ is just $\underline{M_n(A)}$ if $r = 0$ or $s = 0$, and that $M_n^{r+s}(A) \cong M_n^{r+s}(k) \# \underline{A} = M_n^{r+s}(k) \otimes \underline{A}$. Note that if $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space, then $\mathrm{End}(V)$ is isomorphic to $M_n^{r+s}(k)$ as a $\mathbb{Z}_2$-graded algebra, where $r = \dim(V_0)$, $s = \dim(V_1)$ and $n = r + s$. If $\mathrm{End}(V)_1 \neq 0$, then $r > 0$ and $s > 0$.

Lemma 3.3. [11, Theorem IV 3.8] *Let A be a* CSGA *of even type with $A_1 \neq 0$. Suppose A is isomorphic to $M_n(D)$ as an ungraded algebra, where D is a central division algebra over k. Then the following statements hold.*

(a) $Z(A_0) = C_A(A_0)$, *and there exists* $z \in Z(A_0)$, *such that* $Z(A_0) = k \oplus kz$ *and* $z^2 = \alpha \in k^\times$. *The element z is determined up to a scalar multiple by these properties, and hence the square class of α is uniquely determined.*
(b) *Suppose* $\alpha \in k^{\times 2}$. *Then* $Z(A_0) \cong k \times k$, *and* $A \cong M_n^{r+s}(D)$ *for some* $r > 0$ *and* $s > 0$ *with* $r + s = n$.
(c) *Suppose* $\alpha \notin k^{\times 2}$, *and the field* $Z(A_0) \cong k(\sqrt{\alpha})$ *can be embedded into D. Then there exists a grading on D such that* $A \cong \widetilde{M}_n(D)$. *In this case,* $A_0 \cong M_n(D_0)$ *is a* CSA *over* $Z(A_0)$.
(d) *Suppose* $\alpha \notin k^{\times 2}$, *and the field* $Z(A_0) \cong k(\sqrt{\alpha})$ *cannot be embedded into D. Then* $n = 2m$ *is even, and* $A \cong \underline{M_m(D)} \# (\frac{-\alpha,1}{k})_{\mathbb{Z}_2}$ *as graded algebras, where* $(\frac{-\alpha,1}{k})_{\mathbb{Z}_2} = \lfloor k\langle\sqrt{-\alpha}\rangle\rfloor \# \lfloor k\langle\sqrt{1}\rangle\rfloor$ *is a graded quaternion algebra. In this case,* $A_0 \cong M_m(D) \otimes k(\sqrt{\alpha})$ *is a* CSA *over* $Z(A_0)$.

In any case, A_0 is a semisimple k-algebra.

Let A be an H_4-module algebra. Consider the element $h \in H_4$ as a linear endomorphism of A. Then the kernel $\mathrm{Ker}(h)$ of h is an H_4-module subalgebra of A with h acting trivially. We observe that the subalgebra $\mathrm{Ker}(h)$ is a CSGA of even (or odd) type and represents an element in $\mathrm{BW}(k)$ if A is an (H_4, R_0)-Azumaya algebra of odd (or even) type. Now we are ready to give the structure theorem of (H_4, R_0)-Azumaya algebras of odd type.

Theorem 3.4. *Let A be an (H_4, R_0)-Azumaya algebra of odd type. The following assertions hold.*

(a) $Z(A) = C_A(A_0) = k \oplus kz$, *where $z \in Z(A)_1$ and $z^2 = \alpha \in k^\times$. The square class of α does not depend on the choice of $z \in Z(A)_1 \setminus \{0\}$, and $Z(A) \cong k\langle\sqrt{\alpha}\,\rangle$ as graded algebras. Moreover, $A = A_0 \oplus A_0 z$.*

(b) *There exists a unique element $v \in A$ such that*

$$h \cdot a = av - v(g \cdot a), \quad a \in A.$$

Moreover, $v \in A_1$ and $v^2 = \beta \in k$.

(c) *Suppose $\beta \neq 0$. Then we have H_4-module algebra isomorphisms*

$$A \cong \mathrm{Ker}(h) \,\#\, k\big\langle\sqrt{\beta}\big\rangle \quad \text{and} \quad \mathrm{Ker}(h) \cong \big\lfloor A^u_{0(\alpha)} \big\rfloor,$$

where $\lfloor A^u_{0(\alpha)} \rfloor$ is a CSA *over k, $u = vz^{-1} \in A_0$, and $\lfloor A^u_{0(\alpha)} \rfloor$ is a graded algebra defined on A_0 with the grading, the multiplication and the H_4-action given by* (9), (10) *and* (3)*, respectively.*

Proof. (a) It follows from Lemma 3.2(a) and (b).

(b) Let z be a chosen element in part (a). Since z is an invertible element, A is a free (left or right) A_0-module with an A_0-basis $\{1, z\}$. Note that $h \cdot A_0 \subseteq A_1$ and $h \cdot A_1 \subseteq A_0$. Thus, h as a linear endomorphism of A is determined by two k-linear endomorphisms ϕ and ψ of A_0 with

$$h \cdot (x + yz) = \phi(y) + \psi(x)z$$

for all $x, y \in A_0$. Let $a = x + yz$ and $a' = x' + y'z$ be any two elements in A with x, y, x' and y' belonging to A_0. Then from the equation $h \cdot (aa') = (h \cdot a)(g \cdot a') + a(h \cdot a')$, one gets that

$$\phi\big(xy' + yx'\big) = x\phi\big(y'\big) + \alpha y\psi\big(x'\big) + \phi(y)x' - \alpha\psi(x)y' \quad \text{and}$$
$$\psi\big(xx' + \alpha yy'\big) = y\phi\big(y'\big) + x\psi\big(x'\big) - \phi(y)y' + \psi(x)x';$$

from the equation $h^2 \cdot a = 0$, one gets $(\phi\psi)(x) = 0$ and $(\psi\phi)(y) = 0$. Thus we obtain the following equations:

$$\phi(xy) = x\phi(y) - \alpha\psi(x)y, \tag{i}$$

$$\phi(xy) = \alpha x\psi(y) + \phi(x)y, \tag{ii}$$

$$\psi(xy) = x\psi(y) + \psi(x)y, \tag{iii}$$

$$\alpha\psi(xy) = x\phi(y) - \phi(x)y, \tag{iv}$$

$$(\phi\psi)(x) = (\psi\phi)(x) = 0, \tag{v}$$

for all x, $y \in A_0$. By Eq. (iii) we know that ψ is a derivation of the algebra A_0. Since A_0 is a CSA over k, it follows from [14] that ψ is inner, i.e., there exists an element $u_1 \in A_0$ such that

$$\psi(x) = xu_1 - u_1x, \quad \text{for all } x \in A_0.$$

Letting $y = 1$ in Eq. (i), one obtains

$$\begin{aligned}\phi(x) &= x\phi(1) - \alpha\psi(x) = x\phi(1) - \alpha(xu_1 - u_1x)\\ &= \alpha\big[x\big(\alpha^{-1}\phi(1) - u_1\big) + u_1x\big] = \alpha(xu_2 + u_1x)\end{aligned}$$

for all $x \in A_0$, where $u_2 = \alpha^{-1}\phi(1) - u_1 \in A_0$. Similarly, letting $x = 1$ in Eq. (ii), one obtains

$$\phi(y) = \alpha(u_2y + yu_1), \quad \text{for all } y \in A_0.$$

Hence we have

$$\phi(x) = \alpha(xu_2 + u_1x) = \alpha(u_2x + xu_1), \quad \text{for all } x \in A_0.$$

Since $\alpha \neq 0$, we obtain that $(u_1 - u_2)x = x(u_1 - u_2)$ for all $x \in A_0$. Since A_0 is a CSA over k, the element $u_1 - u_2$ is an element γ in k. Let $u = u_1 - \frac{1}{2}\gamma = \frac{1}{2}(u_1 + u_2)$. Then $u \in A_0$, $u_2 = u - \frac{1}{2}\gamma$ and $u_1 = u + \frac{1}{2}\gamma$. Thus we have

$$\phi(x) = \alpha(xu_2 + u_1x) = \alpha\left[x\left(u - \frac{1}{2}\gamma\right) + \left(u + \frac{1}{2}\gamma\right)x\right] = \alpha(xu + ux)$$

and

$$\psi(x) = xu_1 - u_1x = x\left(u + \frac{1}{2}\gamma\right) - \left(u + \frac{1}{2}\gamma\right)x = xu - ux$$

for all $x \in A_0$. Now let $v = uz$. Then $v \in A_1$. Since $z \in Z(A)$ and $z^2 = \alpha$, we have

$$\begin{aligned}h \cdot (x + yz) &= \alpha(yu + uy) + (xu - ux)z\\ &= z\big(y(uz) + (uz)y\big) + \big(x(uz) - (uz)x\big)\\ &= (x + yz)v - v(x - yz)\end{aligned}$$

for all $x, y \in A_0$. This shows that

$$h \cdot a = av - v(g \cdot a), \quad \text{for all } a \in A.$$

Next we show that the element v is unique. If v' is an element in A such that

$$h \cdot a = av' - v'(g \cdot a), \quad \text{for all } a \in A.$$

Then we have $h \cdot z = zv - v(g \cdot z) = 2zv$ and $h \cdot z = zv' - v'(g \cdot z) = 2zv'$. Since $z^2 = \alpha \in k^\times$, z is invertible. It follows that $v' = v$.

Finally, we show that $v^2 \in k$. By Eq. (v), $(\phi\psi)(x) = 0$ for all $x \in A_0$. However, we have

$$(\phi\psi)(x) = \phi(xu - ux) = \alpha\big[(xu - ux)u + u(xu - ux)\big] = \alpha\big(xu^2 - u^2x\big).$$

This implies that $xu^2 - u^2x = 0$ for all $x \in A_0$ as $\alpha \neq 0$. Since A_0 is a CSA over k, $u^2 = \delta \in k$. Thus $v^2 = (uz)^2 = u^2z^2 = \delta\alpha = \beta \in k$.

(c) Suppose $\beta \neq 0$. Then from the proof of part (b) we know that $u = vz^{-1} \in A_0$ and $u^2 = \delta = \alpha^{-1}\beta \neq 0$. Thus A_0 has a $\mathbb{Z}_2$-graded algebra structure A_0^u induced by u (see (9)).

Next we show that the element v generates an H_4-Azumaya subalgebra of A. It is easy to see that $B = k + kv$ is a graded subalgebra of A with $B_0 = k$ and $B_1 = kv$. Since $h \cdot v = vv - v(g \cdot v) = 2v^2 = 2\beta \in k$, B is an H_4-module subalgebra of A, and $B \cong k\langle\sqrt{\beta}\,\rangle$ as H_4-module algebras. By Lemma 1.2, B is (H_4, R_0)-Azumaya as $\beta \neq 0$. Now R_0 is a triangular structure of H_4. So we have that ${}^{R_0}C^l_A(B) = {}^{R_0}C^r_A(B) =: \widehat{C}^{R_0}_A(B)$ is an (H_4, R_0)-Azumaya module subalgebra of A and $\widehat{C}^{R_0}_A(B) \# B \cong B \# \widehat{C}^{R_0}_A(B) \cong A$ by Corollary 2.4.

Now we work out the structure of $\widehat{C}^{R_0}_A(B)$. Note that $A = A_0 \oplus A_0z$ and $z \in Z(A)$. Given $x, y \in A_0$, it is easy to compute that $x + yz \in \widehat{C}^{R_0}_A(B)$ if and only if $xv = vx$ and $yv = -vy$ if and only if $xu = ux$ and $yu = -uy$, where $u = vz^{-1}$. It follows that $\widehat{C}^{R_0}_A(B) = (A_0^u)_0 \oplus (A_0^u)_1 z$. Since $z \in Z(A)$ and $z^2 = \alpha \in k^\times$, $\widehat{C}^{R_0}_A(B)$ is isomorphic, as an $\mathbb{Z}_2$-graded algebra, to $A^u_{0(\alpha)}$.

Next we show that $\widehat{C}^{R_0}_A(B) = \mathrm{Ker}(h)$ so that h acts on $\widehat{C}^{R_0}_A(B)$ trivially. For any element $a = x + yz \in A$ with $x, y \in A_0$. We have

$$\begin{aligned} h \cdot a &= av - v(g \cdot a) = (x + yz)uz - uz(x - yz) \\ &= \alpha(yu + uy) + (xu - ux)z. \end{aligned}$$

Thus, $h \cdot a = 0$ if and only if $xu = ux$ and $yu = -uy$. This shows that $\widehat{C}^{R_0}_A(B) = \mathrm{Ker}(h)$ as H_4-module algebras.

Finally, if $(A_0^u)_1 = 0$, then $A^u_{0(\alpha)} = A_0$ is a CSA over k. If $(A_0^u)_1 \neq 0$, then $A^u_{0(\alpha)}$ is a CSGA of even type since both A and $k\langle\sqrt{\beta}\,\rangle$ are of odd type. In this case, $A^u_{0(\alpha)}$ is a CSA over k as well. $\square$

We remark that the case where $\beta = 0$ not discussed in Theorem 3.4 is because in this case *the action of h is irrelevant*. For if an H_4-Azumaya algebra A with the action of h

induced by an element $v \in A_1$ with $v^2 = 0$, then A is Brauer equivalent to $\lfloor A \rfloor$ in the equivariant Brauer group $\mathrm{BM}(k, H_4, R_0)$ (see the proof of Theorem 3.8). In this sense, A represents an element in the Brauer–Wall group. The same situation occurs for H_4-Azumaya algebras of even type.

In order to determine the structure of an (H_4, R_0)-Azumaya algebra of even type, we need the following result which might be well known. However, for the sake of completeness we include an elementary proof here.

Lemma 3.5. *Assume that A is a finite-dimensional algebra over k. Let n and m be two positive integers. If two matrices $X \in M_{n\times m}(A)$ and $Y \in M_{m\times n}(A)$ satisfy $XY = I_n$ and $YX = I_m$, then $n = m$. Here I_n and I_m are the $n \times n$ and $m \times m$ identity matrices, respectively.*

Proof. We may assume $n \geqslant m$. Let $X = (x_{ij})_{n\times m}$ and $Y = (y_{ij})_{m\times n}$ with $x_{ij}, y_{ij} \in A$. Form the matrices

$$\overline{X} := (X\ 0)_{n\times n} := \begin{pmatrix} x_{11} & \cdots & x_{1m} & 0 & \cdots & 0 \\ x_{21} & \cdots & x_{2m} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{nm} & 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$

and

$$\overline{Y} := \begin{pmatrix} Y \\ 0 \end{pmatrix}_{n\times n} := \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}.$$

Then $\overline{X}$ and $\overline{Y}$ are two elements in $M_n(A)$, the $n \times n$-matrix algebra over A. Obviously, $\overline{X}\,\overline{Y} = I_n$. Recall that in a finite-dimensional algebra over a field, an element is a left unit if and only if it is a right unit if and only if it is invertible. Now since $M_n(A)$ is a finite-dimensional algebra over k, $\overline{X}$ is an invertible matrix in $M_n(A)$ and so $\overline{Y}\,\overline{X} = I_n$. On the other hand, one can see that $\overline{Y}\,\overline{X} = \left(\begin{smallmatrix} I_m & 0 \\ 0 & 0 \end{smallmatrix}\right)$. It follows that $n = m$. $\square$

Recall from [1] that an action of a Hopf algebra H on an algebra A is said to be *strongly inner* if there exists an algebra map $\phi : H \to A$ such that

$$h \cdot a = \sum \phi(h_{(1)}) a \phi\big(S(h_{(2)})\big), \quad \forall h \in H,\ \forall a \in A.$$

Now we consider the structure of an (H_4, R_0)-Azumaya algebra of even type.

Theorem 3.6. *Let A be an (H_4, R_0)-Azumaya algebra of even type with $A_1 \neq 0$. Suppose A is isomorphic to $M_n(D)$ as an algebra, where D is a central division algebra over k. Then the following statements hold.*

(a) *$Z(A_0)=C_A(A_0)$, and there exist $z\in Z(A_0)$ and a unique $v\in A_1$ such that $z^2=\alpha\in k^\times$, $v^2=\beta\in k$, $Z(A_0)=k\oplus kz$, and*

$$g\cdot a=zaz^{-1},\qquad h\cdot a=av-v(g\cdot a),\quad \forall a\in A.$$

Moreover, the element z is determined up to a scalar multiple by these properties, and hence the square class of α is uniquely determined.

(b) *Assume $\beta\neq 0$. Then we have H_4-module algebra isomorphisms*

$$A\cong \operatorname{Ker}(h)\,\#\,k\langle\sqrt{\beta}\rangle\quad \textit{and}\quad \operatorname{Ker}(h)\cong\lfloor \underline{C_{A_0}(v)}\otimes k\langle\sqrt{-\alpha\beta}\,\rangle\rfloor,$$

where $\underline{C_{A_0}(v)}$ is a CSA *over k, v,α and β are elements as above.*

(c) *Suppose $\alpha\in k^{\times 2}$ and $\beta\neq 0$. Then $n=2r$ is even, $C_{A_0}(v)\cong M_r(D)$ and there are H_4-module algebra isomorphisms*

$$A\cong\lfloor \underline{M_r(D)}\otimes k\langle\sqrt{-\beta}\,\rangle\rfloor\,\#\,k\langle\sqrt{\beta}\rangle.$$

(d) *Suppose $\alpha\in k^{\times 2}$ and $\beta=0$. Then $A\cong M_n^{r+s}(D)$ as $\mathbb{Z}_2$-graded algebras for some $r>0$ and $s>0$ with $r+s=n$, and there exist $X\in M_{r\times s}(D)$ and $Y\in M_{s\times r}(D)$ with $XY=0$ and $YX=0$ such that v is mapped to $\left(\begin{smallmatrix}0&X\\Y&0\end{smallmatrix}\right)$ under the above isomorphism. Moreover, the H_4-action on A is strongly inner.*

Proof. (a) By the Skolem–Noether theorem (see [10,14]), the H_4-action on A is inner. Hence, there exist elements z and v' in A with $z^2=\alpha\in k^\times$, $v'^2=\beta'\in k$ and $zv'+v'z=\gamma\in k$ such that

$$g\cdot a=zaz^{-1},\qquad h\cdot a=av'-v'(g\cdot a)$$

for all $a\in A$ (see [20]). Since $za=az$ for all $a\in A_0$ and $A_1\neq 0$, we have $z\in C_A(A_0)$ and $z\notin k$. It follows from Lemma 3.3(a) that $Z(A_0)=C_A(A_0)=k\oplus kz$, where the element z is uniquely determined up to a scalar multiple by the above properties, consequently the square class of α is uniquely determined.

Now let $v'=v_0+v$ with $v_0\in A_0$ and $v\in A_1$. We show that the action of h is in fact determined by the homogeneous element v. We have for any $a\in A$,

$$h\cdot a=av'-v'(g\cdot a)=av_0-v_0(g\cdot a)+av-v(g\cdot a).$$

If $a\in A_0$, then $av_0-v_0(g\cdot a)\in A_0$ and $av-v(g\cdot a)\in A_1$. However, $h\cdot A_0\subseteq A_1$. It follows that $av_0-v_0(g\cdot a)=av_0-v_0a=0$ for all $a\in A_0$. Similarly, one can show that $av_0-v_0(g\cdot a)=av_0+v_0a=0$ for all $a\in A_1$. Thus one obtains

$$h\cdot a=av-v(g\cdot a),\quad \text{for all } a\in A.$$

Next, we show that $v^2\in k$. From the equation $av_0-v_0a=0$ for all $a\in A_0$, one gets $v_0\in Z(A_0)$, and hence $v_0=\theta+\delta z$ for some $\theta,\delta\in k$. Then from the equation $av_0+v_0a=0$

for all $a \in A_1 \neq 0$, one gets $\theta = 0$. Hence $v' = \delta z + v$. Since $v \in A_1$, we have $zv + vz = 0$. It follows that $\beta' = v'^2 = (\delta z + v)^2 = \delta^2 z^2 + v^2 = \alpha\delta^2 + v^2$, and hence $v^2 = \beta' - \alpha\delta^2 = \beta \in k$.

Finally, we show that the element v is unique. Assume that v_1 is an element in A_1 such that $h \cdot a = av_1 - v_1(g \cdot a)$ for all $a \in A$. Then $av - v(g \cdot a) = av_1 - v_1(g \cdot a)$, and hence $a(v - v_1) = (v - v_1)(g \cdot a)$ for all $a \in A$. In particular, $a(v - v_1) = (v - v_1)a$ for all $a \in A_0$. This implies that $v - v_1 \in C_A(A_0) = Z(A_0)$. Thus $v - v_1 \in A_0 \cap A_1 = 0$, and so $v_1 = v$.

(b) Suppose $\beta \neq 0$. Then the subalgebra B of A generated by the element v in part (a) is an H_4-module subalgebra. By a similar argument to the one in Theorem 3.4(c), one obtains that $B = k + kv \cong k\langle\sqrt{\beta}\rangle$ as H_4-module algebras. Hence B is (H_4, R_0)-Azumaya by Lemma 1.2, and $A \cong \widehat{C}_A^{R_0}(B) \# B \cong B \# \widehat{C}_A^{R_0}(B)$ as H_4-module algebras by Corollary 2.4. Note that $\widehat{C}_A^{R_0}(B)$ is of odd type since B is of odd type and A is of even type.

Next we work out the structure of the H_4-Azumaya subalgebra $\widehat{C}_A^{R_0}(B)$. Observe that $A = A_0 \oplus A_0 v$ since v is invertible. Let $a + bv \in A$ with $a, b \in A_0$. It is straightforward to compute that $a + bv \in \widehat{C}_A^{R_0}(B)$ if and only if $av = va$ and $bv = -vb$. Thus $a \in C_{A_0}(v)$ and $b = (\alpha^{-1}bz)z \in C_{A_0}(v)z$. So we obtain that $\widehat{C}_A^{R_0}(B) = C_{A_0}(v) \oplus C_{A_0}(v)zv$. Moreover, the fact that $\widehat{C}_A^{R_0}(B)$ is of odd type infers that $C_{A_0}(v)$ is a CSA over k by Lemma 3.2(b); and that the following $\mathbb{Z}_2$-graded algebra isomorphisms hold:

$$\widehat{C}_A^{R_0}(B) \cong \underline{C_{A_0}(v)} \# k\langle\sqrt{-\alpha\beta}\rangle \cong \underline{C_{A_0}(v)} \otimes k\langle\sqrt{-\alpha\beta}\rangle.$$

Finally, the fact that $\mathrm{Ker}(h) = \widehat{C}_A^{R_0}(B)$ follows from the necessary and sufficient condition:

$$h \cdot (a + bv) = 0 \quad \Leftrightarrow \quad av = va \quad \text{and} \quad bv = -vb, \quad \forall (a + bv) \in A.$$

Thus, we obtain the following H_4-module algebra isomorphisms:

$$A \cong \lfloor \underline{C_{A_0}(v)} \otimes k\langle\sqrt{-\alpha\beta}\rangle \rfloor \# k\langle\sqrt{\beta}\rangle.$$

(c) and (d) Suppose $\alpha \in k^{\times 2}$. By Lemma 3.3(b), we may identify A with the graded algebra $M_n^{r+s}(D)$ for some $r > 0$ and $s > 0$ with $r + s = n$. By part (a), we may assume that $\alpha = 1$, $z = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}$ and $v = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$ with $X \in M_{r\times s}(D)$ and $Y \in M_{s\times r}(D)$. From the equation $v^2 = \beta$, one gets that $XY = \beta I_r$ and $YX = \beta I_s$.

If $\beta = 0$ then $XY = 0$ and $YX = 0$. By [20, Lemma 1], the H_4-action on A is strongly inner. Hence part (d) follows.

Now assume $\beta \neq 0$. It follows from Lemma 3.5 that $r = s$ and $Y = \beta X^{-1}$. Hence $n = 2r$ and $v = \begin{pmatrix} 0 & X \\ \beta X^{-1} & 0 \end{pmatrix}$. Let $a = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \in A_0$ with $U, V \in M_r(D)$. Then $av = va$ if and only if $V = X^{-1}UX$. Hence we have

$$C_{A_0}(v) = \left\{ \begin{pmatrix} U & 0 \\ 0 & X^{-1}UX \end{pmatrix} \,\middle|\, U \in M_r(D) \right\} \cong M_r(D).$$

This completes the proof of part (c). $\square$

From both Theorem 3.4 and Theorem 3.6, we see that the H_4-action on an H_4-Azumaya algebra A is determined by two elements $z \in Z(A)$ (or $Z(A_0)$) and $v \in A_1$. In particular, the element v that determines the action of h on A is unique. Consequently, the constant $\beta = v^2$ is uniquely determined by the H_4-action on A. Thus, we obtain *an invariant* $\beta_A \in k$ for each H_4-Azumaya algebra A. We show that this invariant is stable under Brauer equivalence; and it therefore gives us a group homomorphism from the equivariant Brauer group $\mathrm{BM}(k, H_4, R_0)$ onto the additive group k^+.

Lemma 3.7. *Assume that A and B are two H_4-module algebras. If $v_A \in A_1$ and $v_B \in B_1$ are two elements such that $v_A^2 = \alpha \in k$, $v_B^2 = \beta \in k$, and*

$$h \cdot a = av_A - v_A(g \cdot a), \qquad h \cdot b = bv_B - v_B(g \cdot b), \quad \forall a \in A,\ b \in B,$$

then we have $v = v_A \# 1 + 1 \# v_B \in (A \# B)_1$, $v^2 = \alpha + \beta$ and

$$h \cdot x = xv - v(g \cdot x), \quad \forall x \in A \# B.$$

Proof. Since $v_A \in A_1$ and $v_B \in B_1$, $v \in (A \# B)_1$ and $v^2 = (v_A \# 1 + 1 \# v_B)^2 = v_A^2 \# 1 + 1 \# v_B^2 + v_A \# v_B + (1 \# v_B)(v_A \# 1) = \alpha + \beta \in k$. Let $a \in A$ and $b \in B$. Then we have

$$\begin{aligned} h \cdot (a \# b) &= (h \cdot a) \# (g \cdot b) + a \# (h \cdot b) \\ &= \big(av_A - v_A(g \cdot a)\big) \# (g \cdot b) + a \# \big(bv_B - v_B(g \cdot b)\big) \\ &= \big(av_A \# (g \cdot b) + a \# bv_B\big) - \big(v_A(g \cdot a) \# (g \cdot b) + a \# v_B(g \cdot b)\big) \\ &= (a \# b)(v_A \# 1 + 1 \# v_B) - (v_A \# 1 + 1 \# v_B)\big((g \cdot a) \# (g \cdot b)\big) \\ &= (a \# b)(v_A \# 1 + 1 \# v_B) - (v_A \# 1 + 1 \# v_B)\big(g \cdot (a \# b)\big). \qquad \square \end{aligned}$$

From Lemma 3.7, we see that the invariant β satisfies that $\beta_{A\#B} = \beta_A + \beta_B$ for any two H_4-Azumaya algebras A and B. If A and B are Brauer equivalent, then there exists an H_4-module algebra isomorphism

$$A \# \mathrm{End}(M) \cong B \# \mathrm{End}(N)$$

for two finite-dimensional H_4-modules M and N. Note that the elementary H_4-Azumaya algebra $\mathrm{End}(M)$ is of even type with $\mathrm{End}(M)_1 = 0$ or fitting into part (d) of Theorem 3.6. In any case, we have $\beta_{\mathrm{End}(M)} = 0$ as the H_4-action on it is strongly inner. By Lemma 3.7, we have $\beta_A = \beta_B$. Thus we have obtained a group homomorphism, denoted by β_0 as referred to the triangular structure R_0:

$$\beta_0 : \mathrm{BM}(K, H_4, R_0) \to k^+, \qquad \beta_0\big([A]\big) = \beta_A. \tag{11}$$

Recall that the Brauer–Wall group $\mathrm{BW}(k)$ of k can be regarded as a subgroup of $\mathrm{BM}(k, H_4, R_0)$ in a natural way. That is,

$$\mathrm{BW}(k) = \big\{[A] \in \mathrm{BM}(k, H_4, R_0) \mid h \cdot A = 0\big\}.$$

We have the following short exact sequence that gives the reverse exact sequence of [20, Theorem 8].

Theorem 3.8. *The following short sequence of groups is exact and split*:

$$1 \to \mathrm{BW}(k) \hookrightarrow \mathrm{BM}(k, H_4, R_0) \xrightarrow{\beta_0} k^+ \to 0. \tag{12}$$

Proof. It is enough to prove $\operatorname{Ker}\beta_0 = \mathrm{BW}(k)$ as the inclusion map $\mathrm{BW}(k) \hookrightarrow \mathrm{BM}(k, H_4, R_0)$ is split by the restriction map, i.e., forgetting the action of the element $h \in H_4$. It is not difficult to see that $\mathrm{BW}(k) \subseteq \operatorname{Ker}(\beta_0)$. Now assume that A is an (H_4, R_0)-Azumaya algebra with $\beta_0([A]) = 0$. Then there is a unique element $v \in A_1$ such that

$$h \cdot a = av - v(g \cdot a), \quad a \in A,$$

and $v^2 = 0$. We claim that A and $\lfloor A \rfloor$ are Brauer equivalent, i.e., $[A] = [\lfloor A \rfloor]$ in $\mathrm{BM}(k, H_4, R_0)$. It is sufficient to show that $\lfloor A \rfloor \# \overline{A}$ is an elementary H_4-Azumaya algebra. By definition we have $A \# \overline{A} \cong \mathrm{End}(A)$ as H_4-module algebra. Forgetting the action of h on both sides, we have an $\mathbb{Z}_2$-graded algebra isomorphism: $\lfloor A \rfloor \# \overline{A} \cong \mathrm{End}(A)$. Thus, there is an element $u \in \lfloor A \rfloor \# \overline{A}$ such that $u^2 = 1$ and

$$g \cdot x = uxu^{-1}, \quad x \in \lfloor A \rfloor \# \overline{A}.$$

On the other hand, we have $-\bar{v} \in \overline{A}_1$, and

$$h \cdot \bar{a} = \bar{a}(-\bar{v}) - (-\bar{v})(g \cdot \bar{a}), \quad \bar{a} \in \overline{A}.$$

It follows from Lemma 3.7 that the element $v' = -1 \# \bar{v} \in (\lfloor A \rfloor \# \overline{A})_1$, with $v'^2 = 0$, induces the action of h on $\lfloor A \rfloor \# \overline{A}$, i.e.,

$$h \cdot x = xv' - v'(g \cdot x), \quad x \in \lfloor A \rfloor \# \overline{A}.$$

Thus, the H_4-action on $\lfloor A \rfloor \# \overline{A}$ induced by the algebra map $\phi : H_4 \to \lfloor A \rfloor \# \overline{A}$ sending g and h to u and $u^{-1}v' = uv'$, respectively, is strongly inner. Hence the (H_4, R_0)-Azumaya algebra $\lfloor A \rfloor \# \overline{A}$ is elementary. So $[A] = [\lfloor A \rfloor] \in \mathrm{BW}(k)$. It follows that $\operatorname{Ker}(\beta_0) \subseteq \mathrm{BW}(k)$, and hence $\operatorname{Ker}(\beta_0) = \mathrm{BW}(k)$. □

Remark 3.9. In [20], the additive group k^+ was represented by quaternion algebras. It is showed that the following map:

$$\Phi : k^+ \to \mathrm{BM}(k, H_4, R_0), \qquad \Phi(0) = [k],$$
$$\Phi(\alpha) = \left[k\left\langle\sqrt{\alpha^{-1}}; 0\right\rangle \# k\left\langle\sqrt{-\alpha^{-1}}; 1\right\rangle\right], \quad \alpha \neq 0.$$

is a group monomorphism (see [20, Proposition 7]). The reader may take a while to check that the composite map $\beta_0\Phi(\alpha) = -\frac{1}{4}\alpha$ for any $\alpha \in k^+$. So the group map $-4\beta_0 : \mathrm{BM}(k, H_4, R_0) \to k^+, [A] \mapsto -4\beta_0([A])$ splits the group map Φ in [20, Proposition 7], where β_0 is the group map given by (11). However, the group map β_0 is more natural in the sense that it gives an invariant of each H_4-Azumaya algebra; and it can be generalized to the case where k is a commutative ring. In this sense, we are able to compute the equivariant Brauer group $\mathrm{BM}(k, H_4, R_0)$ for a commutative ring k with $\mathrm{ch}(k) \neq 2$. Another noticeable point is that the invariant β_0 can be generalized to the case where multiple differentials are defined on graded Azumaya algebras, or $E(n)$-Azumaya algebras. The equivariant Brauer group $\mathrm{BM}(k, E(n), R_0)$ has been computed by G. Carnovale and J. Cuadra in [2].

Let n be a natural number. Recall from [16] that the Hopf algebra $E(n)$ is generated, as an algebra, by g and h_i for $i = 1, \dots, n$ with relations

$$g^2 = 1, \qquad gh_i + h_i g = 0, \qquad h_i h_j + h_j h_i = 0, \quad \forall 1 \leqslant i, j \leqslant n.$$

The comultiplication and the antipode are given by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \qquad \Delta(h_i) = h_i \otimes g + 1 \otimes h_i, \quad \varepsilon(h_i) = 0,$$

and

$$S(g) = g^{-1} = g, \qquad S(h_i) = gh_i,$$

where $1 \leqslant i \leqslant n$. $E(n)$ has a triangular structure

$$R_0 = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g).$$

For any $1 \leqslant i \leqslant n$, let E_i be the subalgebra of $E(n)$ generated by g and h_i. It is easy to see that E_i is a Hopf subalgebra of $E(n)$, which is isomorphic to H_4.

Let A be an $E(n)$-module algebra. Then A is a graded algebra with the grading induced by g, i.e., $A_i = \{a \in A \mid g \cdot a = (-1)^i a\}$, $i = 0, 1$. In this case, the braided product $A \#_{R_0} B$ of two $E(n)$-module algebras A and B is exactly the graded product given by (2). Hence A is $(E(n), R_0)$-Azumaya if and only if A is a CSGA.

Now let A be an $(E(n), R_0)$-Azumaya algebra. By Theorems 3.4 and 3.6, there exist unique elements $v_1, \dots, v_n \in A_1$ such that

$$h_i \cdot a = av_i - v_i(g \cdot a), \quad a \in A,\ 1 \leqslant i \leqslant n.$$

Moreover, $v_i^2 = \alpha_i \in k$ for all $1 \leqslant i \leqslant n$. Since $(h_i h_j + h_j h_i) \cdot a = 0$ for all $a \in A$, one gets that $v_i v_j + v_j v_i \in Z(A)$. It is clear that $v_i v_j + v_j v_i \in A_0$, and hence in $Z(A)_0$. It follows from Lemmas 3.2 and 3.3 that $v_i v_j + v_j v_i \in k$ for all $1 \leqslant i, j \leqslant n$. Let $\alpha_{ij} = v_i v_j + v_j v_i \in k$ for all $1 \leqslant i, j \leqslant n$. Then (α_{ij}) is a symmetric $n \times n$-matrix over k with $\alpha_{ii} = 2\alpha_i$. Denote by $\beta_0(A)$ the symmetric matrix (α_{ij}). Using the same argument we did for H_4 we obtain that β_0 induces a group homomorphism, still denoted β_0:

$$\mathrm{BM}\big(k, E(n), R_0\big) \to \mathrm{Sym}_n(k), \qquad [A] \mapsto (\alpha_{ij}),$$

where $\mathrm{Sym}_n(k)$ denote the additive group of $n \times n$ symmetric matrices over k. We summarize the above discussion as follows.

Corollary 3.10. *Let A be an $(E(n), R_0)$-Azumaya algebra. Then there exist unique elements $v_1, \dots, v_n \in A_1$ such that*

$$h_i \cdot a = av_i - v_i(g \cdot a), \quad a \in A,\ 1 \leqslant i \leqslant n.$$

Moreover, $v_i v_j + v_j v_i = \alpha_{ij} \in k$ for all $1 \leqslant i, j \leqslant n$.

The information Corollary 3.10 gives us is that every $(E(n), R_0)$-Azumaya algebra has an invariant, a symmetric matrix. Equivalent $(E(n), R_0)$-Azumaya algebras share the same symmetric matrix invariant. In [2] and [20] (the special case), only a central simple $(E(n), R_0)$-Azumaya algebra has such a symmetric matrix invariant because the Skolem–Noether theorem cannot apply to noncentral simple algebras. Thus our structure Theorems 3.4 and 3.6 can take over the role of the Skolem–Noether theorem in the study of the noncentral simple $(E(n), R_0)$-Azumaya algebras.

As a result of Corollary 3.10, we get the reverse exact sequence of [2, Theorem 4.3] and consequently a Brauer group isomorphism $\mathrm{BM}(k, E(n), R_0) \cong \mathrm{BW}(k) \times \mathrm{Sym}_n(k)$ obtained in [2, Theorem 4.4].

Corollary 3.11. *The following group sequence is exact and split.*

$$1 \to \mathrm{BW}(k) \to \mathrm{BM}\big(k, E(n), R_0\big) \xrightarrow{\beta_0} \mathrm{Sym}_n(k) \to 0,$$

where $\beta_0([A]) = (\alpha_{ij})$, and α_{ij} are given in Corollary 3.10, $1 \leqslant i, j \leqslant n$.

Proof. The proof of the equality $\mathrm{BW}(k) = \mathrm{Ker}(\beta_0)$ is similar to the one in the proof of Theorem 3.8 where $n = 1$. The inclusion map $\mathrm{BW}(k) \hookrightarrow \mathrm{BM}(k, E(n), R_0)$ is split by the restriction map. It remains to be show that β_0 is surjective.

Let $M = (\alpha_{ij})$ be a nonzero symmetric matrix, then there is an invertible matrix C such that $CMC^{\mathrm{tr}} = \mathrm{diag}\{\delta_1, \dots, \delta_r, 0, \dots, 0\}$ is diagonal, where $\delta_i \neq 0$, $i = 1, \dots, r$, r is the rank of M, and C^{tr} denotes the transpose matrix of C. For any $1 \leqslant i \leqslant r$, let $A_i = k\langle\sqrt{\delta_i}\,\rangle$ be the $E(n)$-module algebra on which h_i acts as h does in (4) and h_j-acts trivially if $j \neq i$. Then $A = A_1 \# A_2 \# \cdots \# A_r$ is $(E(n), R_0)$-Azumaya with $\beta_0(A) = CMC^{\mathrm{tr}}$. Now let $C^{-1} = (c_{ij})$, the inverse matrix of C. Then the invertible matrix C^{-1} induces a Hopf algebra automorphism ϕ of $E(n)$ given by $\phi(g) = g$ and $\phi(h_i) = \sum_{j=1}^n c_{ij} h_j$, $i = 1, \dots, n$. Using the automorphism ϕ, we define a new action of $E(n)$ on A: $h \cdot_\phi a := \phi(h) \cdot a$, $h \in E(n)$, $a \in A$. Then the new $E(n)$-module algebra, denoted by ${}_\phi A$, is $(E(n), R_0)$-Azumaya and satisfies $\beta_0({}_\phi A) = M$. So β_0 is surjective. $\square$

Although we know that every $(E(n), R_0)$-Azumaya algebra has a symmetric matrix invariant, the structure of an $(E(n), R_0)$-Azumaya algebra does not seem clear to us. It would be interesting to know the structure theorems of $(E(n), R_0)$-Azumaya algebras.

4. Structure theorems of (H_4, R_t)-Azumaya algebras

In this section, we work out the structure theorems of H_4-Azumaya algebras with respect to triangular structures R_t, where $t \neq 0$. Two structure theorems analogous to Theorems 3.4 and 3.6 will be obtained. The key tool we will use in this section is the braided monoidal equivalence from ${}_{H_4}\mathcal{M}^{R_t}$ to ${}_{H_4}\mathcal{M}^{R_0}$ induced by the lazy dual two cocycle (or Drinfeld twist) θ_t given in Section 1.5.

Throughout this section, we fix an element $t \neq 0$ in k and the corresponding triangular structure R_t of H_4. From Sections 1.4 and 1.5, we know that the lazy dual two cocycle $\theta_t = 1 \otimes 1 + \frac{t}{2}(h \otimes gh)$ induces a braided monoidal equivalence from ${}_{H_4}\mathcal{M}^{R_t}$ to ${}_{H_4}\mathcal{M}^{R_0}$. If A is an H_4-module algebra. Then the H_4-module algebra $\Theta_t(A)$ ($= A$ as an H_4-module) has the multiplication $\bullet$ given by (5):

$$a \bullet b = ab - \frac{t}{2}(h \cdot a)(gh \cdot b)$$

for $a, b \in A$.

For convenience, we let $a^{-1\bullet}$ stand for the inverse of an invertible element a in $\Theta_t(A)$, and let $a^{n\bullet}$ denote the n-fold product $a \bullet a \bullet \cdots \bullet a$ of a in $\Theta_t(A)$. The following lemma is clear.

Lemma 4.1. *Let A be a left H_4-module algebra. Then the following statements hold.*

(a) *If $a \in A$ with $h \cdot a = 0$, then $a \bullet b = ab$ and $b \bullet a = ba$ for all $b \in A$.*
(b) *$(h \cdot a) \bullet b = (h \cdot a)b$ and $a \bullet (h \cdot b) = a(h \cdot b)$ for all $a, b \in A$.*
(c) *If $h \cdot A = 0$, then $\Theta_t(A) = A$ as H_4-module algebras.*

Next we consider the center $Z(A)$ of an H_4-module algebra A, which will play an important role in our structure theorems as it did in Section 3. We view an H_4-module algebra A as a $\mathbb{Z}_2$-graded algebra with the grading induced the action of the group-like element g of H_4. The center $Z(A) = Z(A)_0 \oplus Z(A)_1$ is a graded subalgebra of A. For an algebra A and a subset B of A, we introduce *the opposite centralizer* $C_A^-(B)$ of B in A as follows:

$$C_A^-(B) = \{a \in A \mid ab = -ba,\ \forall b \in B\}.$$

It is clear that $C_A^-(B)$ is a subspace of A. If D is another subspace of A, we sometimes write $C_D^-(B)$ for the subspace $D \cap C_A^-(B)$. When $B = \{x\}$, we simply write $C_D^-(x)$ for $C_D^-(\{x\})$.

Lemma 4.2. *Let A be a left H_4-module algebra. The following statements hold*:

(a) *$Z(\Theta_t(A))_1 = Z(A)_1$ as k-vector spaces.*
(b) *$Z(\Theta_t(A)_0) \cap C^-_{\Theta_t(A)}(\Theta_t(A)_1) = Z(A_0) \cap C_A^-(A_1)$ as k-vector spaces.*
(c) *If A is (H_4, R_t)-Azumaya, then we have the following dimension identity*: $\dim(Z(A_0) \cap C_A^-(A_1)) + \dim(Z(A)_1) = 1$.

Proof. (a) Let $a \in Z(A)_1$. Then $a \in \Theta_t(A)_1$. Now for any $b \in \Theta_t(A)$, we have

$$\begin{aligned}
a \bullet b &= ab - \frac{t}{2}(h \cdot a)(gh \cdot b) = ba - \frac{t}{2}h \cdot \bigl(a(h \cdot b)\bigr) \\
&= ba - \frac{t}{2}h \cdot \bigl((h \cdot b)a\bigr) = ba - \frac{t}{2}(h \cdot b)(h \cdot a) \\
&= ba - \frac{t}{2}(h \cdot b)(gh \cdot a) = b \bullet a.
\end{aligned}$$

This shows that $Z(A)_1 \subseteq Z(\Theta_t(A)) \cap \Theta_t(A)_1 = Z(\Theta_t(A))_1$. Since $\Theta_{-t}(\Theta_t(A)) = A$ as H_4-module algebras, we also have $Z(\Theta_t(A))_1 \subseteq Z(A)_1$. Hence $Z(\Theta_t(A))_1 = Z(A)_1$.

(b) We only need to verify the inclusion $Z(A_0) \cap C_A^-(A_1) \subseteq Z(\Theta_t(A)_0) \cap C^-_{\Theta_t(A)}(\Theta_t(A)_1)$ for the same reason as in the proof of part (a). Let $a \in Z(A_0) \cap C_A^-(A_1)$. Then $a \in \Theta_t(A)_0$, and $ab = (g \cdot b)a$ for all $b \in A$. Hence for any $b \in \Theta_t(A) = A$, we have

$$\begin{aligned}
a \bullet b &= ab - \frac{t}{2}(h \cdot a)(gh \cdot b) = ab - \frac{t}{2}h \cdot \bigl(a(h \cdot b)\bigr) \\
&= (g \cdot b)a - \frac{t}{2}h \cdot \bigl((gh \cdot b)a\bigr) = (g \cdot b)a - \frac{t}{2}(gh \cdot b)(h \cdot a) \\
&= (g \cdot b)a - \frac{t}{2}\bigl(h \cdot (g \cdot b)\bigr)(gh \cdot a) = (g \cdot b) \bullet a.
\end{aligned}$$

Thus, $a \in Z(\Theta_t(A)_0) \cap C^-_{\Theta_t(A)}(\Theta_t(A)_1)$, and $Z(A_0) \cap C_A^-(A_1) \subseteq Z(\Theta_t(A)_0) \cap C^-_{\Theta_t(A)}(\Theta_t(A)_1)$ follows.

(c) Assume that A is (H_4, R_t)-Azumaya. Since $R_t^{\theta_t} = R_0$, the algebra $\Theta_t(A)$ is (H_4, R_0)-Azumaya by Lemma 1.1(a). If $A_1 = 0$, then $Z(A)_1 = 0$ and $Z(A_0) \cap C_A^-(A_1) = Z(A)$. Since H_4 acts on A trivially in this case, A is a CSA over k, and hence $\dim(Z(A_0) \cap C_A^-(A_1)) + \dim(Z(A)_1) = \dim(Z(A)) = 1$. In the case where $A_1 \neq 0$, part (c) follows from parts (a) and (b), Theorems 3.4 and 3.6. $\square$

The dimension identity in Lemma 4.2(c) shows that only one of the two subspaces $Z(A)_1$ and $Z(A_0) \cap C_A^-(A_1)$ is nonzero. This fact leads us to the following parity definition of an (H_4, R_t)-Azumaya algebra.

Definition 4.3. Let A be an (H_4, R_t)-Azumaya algebra. If $Z(A)_1 = 0$, A is said to be of *even type*. If $Z(A)_1 \neq 0$, then A is said to be of *odd type*.

Note that Definition 4.3 is consistent with the case where $t = 0$, and is also consistent with the parity of the product algebra. That is, if we let $\partial(A) = 0$ when A is of even type and $\partial(A) = 1$ otherwise, then we have $\partial(A \#_{R_t} B) = \partial(A) + \partial(B)$ modulo 2. The foregoing parity identity follows from Lemma 4.2 and [11, Product Theorem IV 3.12].

Let A be an H_4-module algebra. Like in Section 3, we consider the H_4-module subalgebra $\mathrm{Ker}(h)$, the kernel of h when considered as a linear endomorphism of A. The subalgebra $\mathrm{Ker}(h)$ is a CSGA of even (or odd) type and represents an element in $\mathrm{BW}(k)$ if A is an (H_4, R_t)-Azumaya algebra of odd (or even) type. In this case, $\mathrm{Ker}(h)$ is equal to

$h \cdot A$, the image of h in A. We first consider the structure theorem of (H_4, R_t)-Azumaya algebras of odd type.

Theorem 4.4. *Let A be an (H_4, R_t)-Azumaya algebra of odd type. The following assertions hold.*

(a) $\dim(Z(A)_1) = 1$. *If $0 \neq z \in Z(A)_1$, then $z^2 = \alpha$ for some $\alpha \in k$.*
(b) *Assume $\alpha = 0$ in part* (a). *Then we have an H_4-module algebra isomorphism*

$$A \cong \mathrm{Ker}(h) \#_{R_t} k\langle\sqrt{0}; 1\rangle = \mathrm{Ker}(h) \#_{R_0} k\langle\sqrt{0}; 1\rangle,$$

where $\mathrm{Ker}(h) = C_{A_0}(u) \oplus C^-_{A_1}(u) = h \cdot A$ *is a* CSA *over k and $u = h \cdot z$, where $z \in Z(A)_1$ as in part* (a).
(c) *Suppose $\alpha \neq 0$. Then there exists a unique element $v \in A_1$ such that the action of h is given by the element v, that is,*

$$h \cdot a = av - v(g \cdot a), \quad \forall a \in A.$$

Moreover, $v^2 = \beta \in k$ and $1 - 2t\beta \neq 0$.
(d) *Suppose $\alpha\beta \neq 0$. Then we have an H_4-module algebra isomorphism*

$$A \cong \mathrm{Ker}(h) \#_{R_t} k\langle\sqrt{\beta}\,\rangle = \mathrm{Ker}(h) \#_{R_0} k\langle\sqrt{\beta}\,\rangle,$$

where $\mathrm{Ker}(h) = C_{A_0}(u) \oplus C^-_{A_0}(u)z$ *is a* CSGA *isomorphic to $A^u_{0\,(\alpha)}$ and the element $u = vz \in A_0$ with z and v given in parts* (a) *and* (c)*, respectively.*

Proof. (a) By Lemma 4.2(c), $\dim(Z(A)_1) = 1$. Let $0 \neq z \in Z(A)_1$. By Lemmas 1.1(a) and 4.2(a), the algebra $\Theta_t(A)$ is an (H_4, R_0)-Azumaya algebra of odd type and $z \in Z(\Theta_t(A))_1$. Thus, Theorem 3.4 applies and we obtain that $z^{2\bullet} = \alpha' \in k^\times$. We show that $z^2 \in k$. Again by Theorem 3.4, there exists a unique $v_1 \in \Theta_t(A)_1 = A_1$ such that

$$h \cdot a = a \bullet v_1 - v_1 \bullet (g \cdot a), \quad \text{for all } a \in \Theta_t(A).$$

Moreover, $v_1^{2\bullet} = \beta' \in k$, and hence $h \cdot v_1 = 2\beta'$. Now $h \cdot z = z \bullet v_1 - v_1 \bullet (g \cdot z) = 2z \bullet v_1$. It follows that

$$\begin{aligned} z^{2\bullet} &= z^2 - \frac{t}{2}(h \cdot z)(gh \cdot z) = z^2 - \frac{t}{2}(h \cdot z)(h \cdot z) \\ &= z^2 - \frac{t}{2}(h \cdot z) \bullet (h \cdot z) = z^2 - 2tz^{2\bullet} \bullet v_1^{2\bullet} \\ &= z^2 - 2t\alpha'\beta'. \end{aligned}$$

Hence, $z^2 = \alpha' + 2t\alpha'\beta' = \alpha'(1 + 2t\beta') = \alpha \in k$.

(b) Assume $\alpha = 0$. We use the notations in the proof of part (a). Since $\alpha' \neq 0$, $1 + 2t\beta' = 0$ and $\beta' = -\frac{1}{2}t^{-1}$ as $t \neq 0$. Thus $v_1 \in A_1$ and $h \cdot v_1 = 2\beta' = -t^{-1}$. We also have

$$\begin{aligned}
h\cdot a &= a\bullet v_1 - v_1\bullet(g\cdot a)\\
&= av_1 - v_1(g\cdot a) - \frac{t}{2}\big[(h\cdot a)(gh\cdot v_1) - (h\cdot v_1)(ghg\cdot a)\big]\\
&= av_1 - v_1(g\cdot a) - 2t\beta'(h\cdot a),
\end{aligned}$$

for all $a \in A$. Hence $av_1 - v_1(g\cdot a) = (1+2t\beta')h\cdot a = 0$ for all $a\in A$. Furthermore, one gets that $\beta' = v_1^{2\bullet} = v_1^2 - \frac{t}{2}(h\cdot v_1)(gh\cdot v_1) = v_1^2 - 2t\beta'^2$. This implies that $v_1^2 = \beta' + 2t\beta'^2 = \beta'(1+2t\beta') = 0$.

Now let $B = \operatorname{span}\{1, v_1\}$. Then B is an H_4-module subalgebra of A and $B \cong k\langle\sqrt{0}; -t^{-1}\rangle$ as H_4-module algebras. It follows from Lemmas 1.2(b) and 1.3(b) that B is (H_4, R_t)-Azumaya and $B \cong k\langle\sqrt{0}; 1\rangle$. By Corollary 2.4, $\widehat{C}_A^{R_t}(B)$ is (H_4, R_t)-Azumaya and there are H_4-module algebra isomorphisms

$$A \cong \widehat{C}_A^{R_t}(B)\,\#_{R_t}\, B \cong B\,\#_{R_t}\,\widehat{C}_A^{R_t}(B).$$

Let $a\in A$. Then $a \in \widehat{C}_A^{R_t}(B)$ if and only if $av_1 = \Sigma(R_t^{(2)}\cdot v_1)(R_t^{(1)}\cdot a)$. A straightforward computation shows that $\Sigma(R_t^{(2)}\cdot v_1)(R_t^{(1)}\cdot a) = v_1(g\cdot a) - h\cdot a$. Hence $a\in\widehat{C}_A^{R_t}(B)$ if and only if $h\cdot a = v_1(g\cdot a) - av_1 = 0$ if and only if $a \in \operatorname{Ker}(h)$. It follows that $\widehat{C}_A^{R_t}(B) = \operatorname{Ker}(h)$. Since h acts trivially on $\operatorname{Ker}(h)$, the algebra $\operatorname{Ker}(h)$ is a CSGA over k by Lemma 4.1(c). Now the fact that both A and B are of odd type implies that $\operatorname{Ker}(h)$ is of even type. Hence, $\operatorname{Ker}(h)$ is a CSA over k by Lemma 3.3.

Next, we show that $\operatorname{Ker}(h)$ can be described by the element $u = h\cdot z$, where $z\in Z(A)_1$ is given in part (a). Namely, we show that $\operatorname{Ker}(h) = C_{A_0}(u)\oplus C^-_{A_1}(u)$. For $a\in A$, we have

$$\begin{aligned}
h\cdot(az) &= (h\cdot a)(g\cdot z) + a(h\cdot z) = au - (h\cdot a)z, \quad\text{and}\\
h\cdot(za) &= (h\cdot z)(g\cdot a) + z(h\cdot a) = u(g\cdot a) + z(h\cdot a).
\end{aligned}$$

Since $z\in Z(A)$, it follows from Lemma 4.1(b) that $au - (h\cdot a)\bullet z = u(g\cdot a) + (h\cdot a)\bullet z$. Note that z is an invertible element in $\Theta_t(A)$ though it is not invertible in A. Thus we obtain that $h\cdot a = 0$ if and only if $au = u(g\cdot a)$. This shows that $\operatorname{Ker}(h) = C_{A_0}(u)\oplus C^-_{A_1}(u)$. Note that $u^2 = 4\alpha'\beta' = -2t^{-1}\alpha' = \delta \in k^\times$.

Lastly, we show that $\operatorname{Ker}(h) = h\cdot A = \{h\cdot a \mid a\in A\}$. Indeed, let $a\in\operatorname{Ker}(h)$, then $h\cdot a = 0$ and $au = u(g\cdot a)$. Hence we have

$$\begin{aligned}
a &= u(g\cdot a)u^{-1} = \delta^{-1}u(g\cdot a)u = \delta^{-1}(h\cdot z)(g\cdot a)(h\cdot z)\\
&= \delta^{-1}\big(h\cdot(za)\big)(h\cdot z) = \delta^{-1}\big(h\cdot(za)\big)(gh\cdot z)\\
&= \delta^{-1}h\cdot\big(za(h\cdot z)\big)\in h\cdot A.
\end{aligned}$$

So $\operatorname{Ker}(h)\subseteq h\cdot A$. The other inclusion is obvious.

(c) Suppose $\alpha \neq 0$. By the proof of part (a), we have $\alpha = \alpha'(1+2t\beta')$ and $\alpha' \neq 0$. Thus, $1+2t\beta' \neq 0$. Let $v = (1+2t\beta')^{-1}v_1 \in A_1$, where v_1 is given in the proof of part (a). From the proof of part (b), one can see that $v^2 = \beta'(1+2t\beta')^{-1} \equiv \beta \in k$ and

$$h \cdot a = av - v(g \cdot a), \quad \text{for all } a \in A.$$

Moreover, $1-2t\beta = 1-2t\beta'(1+2t\beta')^{-1} = (1+2t\beta')^{-1} \neq 0$.

Now assume that there is a $v' \in A$ such that

$$h \cdot a = av' - v'(g \cdot a), \quad \text{for all } a \in A.$$

Then we have $h \cdot z = zv - v(g \cdot z) = 2zv$ and $h \cdot z = zv' - v'(g \cdot z) = 2zv'$. Since $z^2 = \alpha \neq 0$, z is invertible. Hence $v' = v$.

(d) Suppose $\alpha \neq 0$ and $\beta \neq 0$. Let v be the element given in part (c) and let $u = zv = \frac{1}{2}h \cdot z$. The subalgebra $B = k + kv$ generated by v is an (H_4, R_t)-Azumaya subalgebra of A isomorphic to $k\langle\sqrt{\beta}\rangle$. For any element $a \in A$, one can easily check that $\Sigma(R_t^{(2)} \cdot v)(R_t^{(1)} \cdot a) = av$ if and only if $(1-2t\beta)(h \cdot a) = 0$, if and only if $h \cdot a = 0$, if and only if $av = v(g \cdot a)$, if and only if $au = u(g \cdot a)$. Hence $\widehat{C}_A^{R_t}(B) = \operatorname{Ker}(h) = C_{A_0}(u) \oplus C^-_{A_1}(u)$.

Since $z \in Z(A)$ and $z^2 = \alpha \in k^\times$, we have $A_1 = A_0 z$. Consequently, $C^-_{A_1}(u) = C^-_{A_0}(u)z$. This implies that $\widehat{C}_A^{R_t}(B) = C_{A_0}(u) \oplus C^-_{A_0}(u)z \cong \lfloor A^u_{0(\alpha)} \rfloor$. Thus, by Corollary 2.4 we have an H_4-module algebra isomorphism

$$A \cong \lfloor A^u_{0(\alpha)} \rfloor \#_{R_t} k\langle\sqrt{\beta}\rangle = \lfloor A^u_{0(\alpha)} \rfloor \#_{R_0} k\langle\sqrt{\beta}\rangle.$$

Moreover, $A^u_{0(\alpha)}$ is a CSA over k since it is a CSGA of even type. $\square$

Finally we consider the structures of (H_4, R_t)-Azumaya algebras of even type with a nontrivial grading. An (H_4, R_t)-Azumaya algebra with the trivial grading is a CSA with the trivial H_4-action. We have the following structure theorem analogous to Theorem 3.6. We will omit the proof as it is quite similar to the proof of Theorem 4.4.

Theorem 4.5. *Let A be an (H_4, R_t)-Azumaya algebra of even type with $A_1 \neq 0$. Then the following statements hold.*

(a) $\dim(Z(A_0) \cap C^-_A(A_1)) = 1$. *Let $0 \neq z \in Z(A_0) \cap C^-_A(A_1)$. Then $z^2 = \alpha \in k$.*
(b) *Assume $\alpha = 0$ in part* (a). *Then there is an H_4-module algebra isomorphism*

$$A \cong \operatorname{Ker}(h) \#_{R_t} k\langle\sqrt{0}; 1\rangle = \operatorname{Ker}(h) \#_{R_0} k\langle\sqrt{0}; 1\rangle.$$

Moreover, $\operatorname{Ker}(h) \cong \lfloor \underline{C_{A_0}(u)} \otimes k\langle\sqrt{\delta}\rangle \rfloor$ *and $C_{A_0}(u)$ is a* CSA *over k, where $u = h \cdot z \in A_1$ with $u^2 = \delta \in k^\times$.*

(c) *Assume $\alpha \neq 0$. Then $g \cdot a = zaz^{-1}$ for all $a \in A$ and there exists a unique element $v \in A_1$ such that*

$$h \cdot a = av - v(g \cdot a), \quad \forall a \in A.$$

Moreover, $v^2 = \beta \in k$ and $1 - 2t\beta \neq 0$.

(d) *Suppose that $\alpha\beta \neq 0$ with α and β given in parts* (a) *and* (c). *Then we have an H_4-module algebra isomorphism*

$$A \cong \operatorname{Ker}(h) \#_{R_t} k\langle\sqrt{\beta}\rangle = \operatorname{Ker}(h) \#_{R_0} k\langle\sqrt{\beta}\rangle.$$

Moreover, $\operatorname{Ker}(h) \cong \lfloor \underline{C_{A_0}(u)} \otimes k\langle\sqrt{-\alpha\beta}\rangle \rfloor$, *and $C_{A_0}(u)$ is a* CSA *over k and the element $u = zv = \frac{1}{2}h \cdot z \in A_1$ with $u^2 = -\alpha\beta \in k^\times$.*

To end this section, we point out that we have an exact sequence similar to (12) for the Brauer group $\mathrm{BM}(k, H_4, R_t)$ when $t \neq 0$. Let A be an (H_4, R_t)-Azumaya algebra. The *invariant map* β_t from $\mathrm{BM}(k, H_4, R_t)$ onto k^+ can be defined in a slightly different manner. Let A be an (H_4, R_t)-Azumaya algebra. A has an one-dimensional graded subspace:

$$C(A) = \big(Z(A_0) \cap C_A^-(A_1)\big) \oplus Z(A)_1.$$

If $C(A)^2 = 0$ (this cannot happen when $t = 0$), we let $\beta_t([A]) = -\frac{1}{2}t^{-1}$. If $C(A)^2 \neq 0$, then by Theorems 4.4 and 4.5, A has a unique element $v \in A_1$ such that $v^2 = \beta \in k$, $1 - 2t\beta \neq 0$ and

$$h \cdot a = av - v(g \cdot a), \quad \text{for all } a \in A.$$

In this case, we define $\beta_t([A]) = \beta(1 - 2t\beta)^{-1}$.

Observe that $\beta_t([A]) = \beta_0([\Theta_t(A)])$ for any (H_4, R_t)-Azumaya algebra A. It follows that β_t is a group homomorphism from $\mathrm{BM}(k, H_4, R_t)$ to k^+ and fits in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{BM}(k, H_4, R_t) \arrow[rr, "\Theta_t"] \arrow[dr, "\beta_t"'] & & \mathrm{BM}(k, H_4, R_0) \arrow[dl, "\beta_0"] \\
& k^+ &
\end{tikzcd}$$

where $\Theta_t : \mathrm{BM}(k, H_4, R_t) \to \mathrm{BM}(k, H_4, R_0)$ is given by $\Theta_t([A]) = [\Theta_t(A)]$. Recall from [21] that $\mathrm{BW}(k)$ can be viewed as a subgroup of $\mathrm{BM}(k, H_4, R_t)$ in a natural way, where $t \in k$. That is,

$$\mathrm{BW}(k) := \big\{[A] \in \mathrm{BM}(k, H_4, R_t) \mid h \cdot A = 0\big\}.$$

It is clear that the group isomorphism Θ_t restricted to $\mathrm{BW}(k)$ becomes the identity map on $\mathrm{BW}(k)$. Thus, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{BW}(k) \arrow[r, hook] \arrow[d, equal] & \mathrm{BM}(k, H_4, R_t) \arrow[r, "\beta_t"] \arrow[d, "\Theta_t"'] & k^+ \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & \mathrm{BW}(k) \arrow[r, hook] & \mathrm{BM}(k, H_4, R_0) \arrow[r, "\beta_0"] & k^+ \arrow[r] & 0
\end{tikzcd}$$

Finally, we point out that the structure of an Azumaya algebra in the Yetter–Drinfeld module category ${}_{H_4}\mathcal{YD}^{H_4}$ is unclear. In [6] we have computed all 4-dimensional Azumaya algebras in ${}_{H_4}\mathcal{YD}^{H_4}$. We found that there is no decomposition for certain Azumaya algebras. Thus, the structure theorems of H_4-Azumaya algebras remain unknown in general.

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